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Tame parametrised chain complexes

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Abstract

Persistent homology has proven to be a useful tool to extract information from data sets. Its method can be summarised by a standard workflow: start with data, build the chain complex of a simplicial complex modelling the data, apply homology obtaining the so-called *persistent module*, and retrieve topological information using invariants. Complete, and thus most discriminative, invariants are given by the indecomposables of the persistent modules. However, such invariants can be retrieved only for the objects of finite representation type whose decomposition is efficiently computed. In addition, homology might be an overkill, and some information may be lost while applying it to the chain complexes. The starting point of our investigation is the idea that a direct study of the chain complex can address these issues. Therefore, we investigate the category of *tame parametrised chain complexes*, which are chain complexes evolving according to one real parameter. Such a category is quite rich and includes many interesting types of objects, such as parametrised vector spaces, commutative ladders and zigzag modules. We define a model category structure on the category of tame parametrised chain complexes. This setting is quite natural since chain complexes admit a model category structure themselves. Moreover, we can exploit the rich theory of model category to extract invariants. In general, in a model category, there are special objects called *cofibrant objects*, that can be used to study any other object in the category by approximating it through them. After identifying the cofibrant objects in the category of tame parametrised chain complexes, we study their indecomposables. We find that, despite in general tame parametrised chain complexes are of wild representation type, the indecomposables of cofibrant objects can be fully described. We then approximate every tame parametrised chain complex using two cofibrant objects, called the *minimal cover* and the *minimal representative.* Such objects are crucial since they are invariants. In particular, the minimal cover is a homological invariant, and the minimal representative is a homotopical invariant. Thus, these two objects are retrieving all the topological information of the objects they are approximating. In conclusion, we prove that it is possible to analyse data using a new workflow: start with data, build the chain complex of a simplicial complex modelling the data, associate to it either a minimal cover or a minimal representative, and decompose the chosen one to retrieve a summary of the information in the data.

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Contents

Notation

$\mathtt{k}$	Field
$\mathbb{N}$	Set of natural numbers
$\mathbf{Ob_C}$	Class of objects of a category $\mathbf{C}$
$\mathbf{Hom_C}$	Class of morphisms of a category $\mathbf{C}$
$\mathbf{Hom_C}(X, Y)$	Subclass of $\mathbf{Hom_C}$ of morphisms of all morphisms $X \to Y$ in $\mathbf{C}$
$\mathbf{Hom_C}(X, -)$, $\mathbf{Hom_C}(-, X)$	Hom functors
$\mathbf{Set}$	Category of sets
$\mathbf{Ab}$	Category of abelian groups
$\mathbf{C^{op}}$	Opposite category of a category $\mathbf{C}$
$\mathbf{fun}(\mathbf{C}, \mathbf{D})$	Category of functors between categories $\mathbf{C}$ and $\mathbf{D}$
$\mathbf{Vect}_\mathtt{k}$	Category of vector spaces
$\mathbf{vect}_\mathtt{k}$	Category of compact vector spaces
$\mathbf{GVect}_\mathtt{k}$	Category of graded vector spaces
$\mathbf{Ch}$	Category of chain complexes
$\mathbf{ch}$	Category of compact chain complexes
$\mathbf{tame}([0, \infty), \mathbf{vect}_\mathtt{k})$	Category of compact parametrised vector spaces
$\mathbf{tame}([0, \infty), \mathbf{ch})$	Category of tame parametrised chain complexes
$\mathbf{ZS}_\mathcal{C}$	Category of zigzag sequences of zigzag profile $\mathcal{C}$
$\mathbf{Zigzag}$	Category of zigzags
$\mathbf{C} \downarrow Y$ $(X \uparrow \mathbf{C})$	(Co)slice category
$[n]$	Poset of the first n natural numbers
$[0, \infty)$	Poset of nonnegative real numbers
$*$	Terminal object
$\emptyset$	Initial object
LF	Left Kan extension of a functor F
SV	Suspension of a graded vector space V
DV	Cone of a graded vector space V
ZX	Graded vector space of the cycles of a chain complex X
BX	Graded vector space of the boundaries of a chain complex X
HX	Graded vector space of the homology of a chain complex X
CX	Cone of a chain complex X

PX	Path of a chain complex X
S^h	h-sphere
D^h	h-disk
$MC(f)$	Mapping cylinder of a chain map f
$\mathbb{I}_{[b,d)}$	Interval vector space
$I^h[b,d)$	Interval sphere of dimension h
$\mathbb{Z}_{[s,e]}$	Interval (chain) zigzag
$\mathbf{1}_X$	Identity morphisms on X
∂	Boundary map
δ	Connecting morphism
$X^{s<t}\colon X^s \to X^t$	Transition morphism of X
$\rightarrowtail$	Monomorphism
$\xrightarrow{\sim}$	Weak equivalence
$\twoheadrightarrow$	Fibration
$\hookrightarrow$	Cofibration
$\hat{f}$	Mediating morphism of the pushout along f

Introduction

Since its first developments, the science of complex systems has been recognised as a very fruitful, multidisciplinary approach to address theoretical problems and related applications in many different fields of research, ranging from traffic flow theory to quantitative biology and social sciences.

Data Science aims at analysing the variety of data produced when studying complex systems and networks by using more and more advanced modelling and computational tools. The massive amount of data, however, often goes beyond the capability of existing computers. Moreover, the effectiveness of standard data analytic tools strongly depends on the accuracy of the measurements used to obtain the data, making such tools often not informative enough. In these respects, it has been soon recognised that improved mathematical tools are necessary to simplify and summarise the features of these huge data sets. The first need is to convert the data into signatures that are less sensitive to noise variations and then feed them into the standard machine learning machine. Algebraic Topology can be used to focus on specific signatures: topological invariants that depend only on global features of simplicial complexes built from the data. These signatures are, by definition, less sensitive to local noise. These considerations prompt the development of a new branch of studies, Topological Data Analysis (TDA), which has found numerous applications in a diverse range of fields [46].

As described by Carlsson [10], TDA provides four major advantages:

- The capability of highlighting particular large-scale behaviours, allowing for the extraction of quantitative information, such as connectivity and presence of loops.
- The possibility of ignoring the precise values of the metric involved, while preserving only the vicinity information. Such metrics, especially in the medical and biological contexts, are often non-intrinsic, and their choice is not always justified theoretically.
- The study of properties which do not depend on the coordinates. Data coordinatisation is often a consequence of the used storage methods and thus does not carry intrinsic information.
- The possibility of studying the whole range of variation of a parameter and analysing the interplay between the geometrical objects constructed from data

while the parameter evolves. This allows not to select a threshold a priori, avoiding the issue of choosing the right cutoff.

One of TDA's most successful tools is persistent homology. The underlying idea traces back to the early 1990s with the works of Frosini [18] and Robins [38]. Persistent homology studies the homology of a dataset, embodying many of the advantages above recalled. Indeed, homology catches global information of a data set, studying its loops in different dimensions. It is called *persistent* because it analyses the features persisting in the data when a parameter changes. One of the most used parameters is the reciprocal distance from the elements of the dataset seen as points in a metric space, and thus it is possible to study vicinity features without introducing coordinates.

Today, persistent homology profits of techniques from different mathematical disciplines, such as Morse theory [31] and quiver representation [33]. Moreover, many algorithms have been developed for persistent homology computation [6, 7], and its applications vary from medicine to material science [8, 34, 42, 43].

Persistent homology method can be described by a standard workflow: start from the data, build a simplicial complex, such as the Vietoris-Rips complex, compute its chain complex describing simplexes adjacency, apply homology obtaining the so-called *persistent module*, and retrieve topological information. Such information is summarised using invariants. Among the various invariants that can be used, the complete ones are given by the indecomposables decomposing persistent modules. Completeness of the invariants is useful because it guarantees the maximal discriminative power. However, it is not always possible to describe the invariants extracted by the indecomposables: one is limited to the objects of finite representation type, for which it is possible to list all the indecomposable types [33], and, among them, only to the objects whose decomposition can be computed algorithmically. For example, a class of objects called *commutative ladders* cannot be analysed using its indecomposables because it is of wild representation type [9]. On the other hand, the indecomposables of another class of objects, the *zigzag modules*, are fully described, but so far there is no efficient software to analyse them [11, 12, 13]. In addition, homology might be an overkill, and some information may be lost while applying it to the chain complexes. Thus, it would be desirable to obtain also homotopical invariants, since homotopy is less forgetful about the shape of data than homology is. Some work has already been done in this direction [19]. However, homotopy theory is typically harder to be turned into a computable tool, and this is the reason why TDA has focused more on homology.

The starting point of our investigation is the idea that a direct study of the chain complex built from data can address these issues. The idea comes from the work of Dwyer and Spaliński [16], where the study of chain complexes as a model category provides information not just about the homology but also about the homotopy of the objects.

Applying the idea of persistence to chain complexes has led us to introduce *tame* *parametrised chain complexes*: the category at the core of this thesis. They are chain complexes evolving according to one real parameter. We call them *tame* since there is only a finite number of values of the parameter at which the chain complexes change. It is reasonable to allow only for a finite number of changes because real data are typically finite. The category of tame parametrised chain complexes is quite rich and includes many interesting types of objects, such as persistent modules, which we call parametrised vector spaces, commutative ladders, and zigzag modules, which we call simply zigzags.

We define a model category structure on the category of tame parametrised chain complexes. This setting is quite natural since we know that chain complexes admit a model category structure themselves [16]. Moreover, we can take advantage of the rich model category theory to extract invariants. In general, in a model category, there are some special objects called *cofibrant objects*, over which one has a little bit more control. These objects can be used to study the other objects in the category by approximating them through cofibrant objects.

Following this standard path in model category theory, we begin with identifying the cofibrant objects in the category of tame parametrised chain complexes. They correspond to chain complexes that grow along with one real parameter, also known in the literature as filtered chain complexes. Such objects are central in the theory of persistent homology since one of the most used simplicial complex is the Vietoris-Rips complex, whose chain complex is filtered. We then study such objects, finding that, despite in general tame parametrised chain complexes are of wild representation type, the indecomposables of cofibrant objects can be fully described [5, 30, 44, 45]. One of the reasons why the decomposition of cofibrant objects is crucial is that it is computable algorithmically [6, 7]. This means that we can use the number and type of the indecomposables as invariants for the cofibrant objects. These invariants are in perfect accordance with the previous theory of persistent homology: by applying homology to the indecomposables of a cofibrant object, we retrieve the decomposition of the persistent module given by the homology of the cofibrant object.

After studying cofibrant objects, we use them to approximate any other tame parametrised chain complex. In general, an object admits many cofibrant approximations but among all of them, two are of particular interest, namely the *minimal cover* and the *minimal representative*. In general, these two objects do not need to exist in a model category. However, whenever they exist, they are invariants. We prove that, in the category of tame parametrised chain complexes, both the minimal cover and the minimal representative exist. The minimal cover is a homological invariant, and the minimal representative is a homotopical invariant. Thus, these two objects are retrieving all the topological information of the objects they are approximating. Since they both are cofibrant, and thus of finite representation type, it is possible to define invariants for any tame parametrised chain complex using the number and type of the indecomposables of its minimal cover and minimal representative.

In conclusion, we prove that it is possible to analyse data using a new workflow: start with data, build the chain complex of a simplicial complex modelling the data, associate to it either a minimal cover or a minimal representative, and decompose the chosen one to retrieve a summary of the information in the data. This method allows us to extract invariants for any tame parametrised chain complexes.

Outline of the thesis. In Chapter 1, we recall the basic definitions and results of category theory. We aim for this work to be as much self-contained as possible. Therefore, we review all the classical results we make use of, providing the pertinent references.

In Chapter 2, we define the categories that are used throughout the whole thesis, along with some essential property of them. Part of these results is already well known, but we decided to prove and include them to comply with our notation. We define the discretisation of the parametrising poset of a functor category. The definition is crucial and is used extensively in the thesis. We next study compactness in the category of interest. Finally, we motivate the study of tame parametrised chain complexes with three examples. In particular, we prove that parametrised vector spaces, commutative ladders, and zigzag can be seen as tame parametrised chain complexes.

In Chapter 3, we show how to extend the model structure of a model category to the category of its tame parametrised objects. Before presenting the result, we introduce the definition and properties of a model category. Here, we decided to include all the proofs to keep the work self-contained. We also provide an explicit example of a model category, the category of compact chain complexes, and we use the model category setting to prove the standard decomposition of compact chain complexes. Finally, we define the distinguished classes of morphisms for model categories of tame parametrised objects, and we prove that they verify the axioms of a model category.

In Chapter 4, we describe the cofibrant objects in the model category of tame parametrised objects of a given model category. Then we prove the decomposition theorem for cofibrant objects in the model category of tame parametrised chain complexes. This decomposition theorem is crucial since it provides invariants for cofibrant objects. Moreover, since the cofibrant objects are used to define invariants for general parametrised chain complexes, such a decomposition provides information about any tame parametrised chain complex.

In Chapter 5, we first study invariants of a general model category, describing the minimal factorisation, the minimal cover and the minimal representative in any model category. We show that if any of them exists, then it is unique up to isomorphisms. This in particular shows that they are invariants. We then proceed to study in details minimality in model categories of tame parametrised objects. In Section 5.2, we show that if in a model category the minimal factorisation and the minimal cover exist, then they do also in the model category of its tame parametrised objects. In Section 5.3, we study the minimal representative and the minimal cover for compact chain complexes, showing that they both exist in the category of compact chain complexes. In Section 5.4, we prove that minimal representatives exist for tame parametrised chain complexes, and we provide a characterisation for both the minimal cover and the minimal representative in such a category. Finally, in Section 5.5, we analyse the minimality in the case of parametrised vector spaces, commutative ladders and zigzags.

Original contributions. In general, the whole approach of the thesis is original [14]. We are not aware of other authors studying tame parametrised chain complexes directly or using the model category theory in connection with TDA. The definition of a model structure on the category of tame parametrised chain complexes, as well as the proof that such a structure satisfies the axioms of a model category, are original results. The cofibrant tame parametrised chain complexes are known in the literature as filtered chain complexes. Even though their decomposition theorem is already known [5, 30, 44, 45], the method we apply to prove such a decomposition is novel. The embedding of different classes of objects in the category of tame parametrised chain complexes is a further original result. While the idea of minimality is not novel in model category theory [39], the results proving the existence of minimal cover and minimal representative in the model category of tame parametrised chain complexes, and their characterisation, are original.

Chapter 1

Preliminaries on category theory

Category theory is the field of mathematics that describes relations between objects, where objects are intended in the most general terms. The definition of a category itself asks no conditions on the objects, but it requires some minimal assumptions on the relations, so-called *morphisms*. These requirements are the bare minimum that can be asked, and often they are not enough to guarantee the existence of all the needed constructions. In these cases, it is necessary to add some additional structure to the category, such as abelianity or a model structure.

Considering relations, one may want to study relations at a higher level than between objects, and investigate the relations between categories. In this case, relations are known under the name of *functors*, and they must satisfy some essential requirements, so to transfer the structure of a category correctly to another one.

Moving to an even higher level of abstraction, one may consider relations between functors. In this case, one talks about *natural transformations*.

In this chapter, we provide the introductory definitions for the study of category theory. The chapter is organised as follows. In Section 1.1, we introduce the concepts of a category and a functor, along with their basic properties. In Section 1.2, we define abelian categories, and the main constructions allowed therein. Finally, in Section 1.3, we focus on an example of an abelian category given by the category of chain complexes.

A classical reference for the notions treated in this chapter is [28].

1.1 Categories and functors

Definition 1.1. A *category* $\mathbf{C}$ is given by:

(i) a class $\mathbf{Ob_C}$ of objects;

(ii) a class $\mathbf{Hom_C}$ of morphisms between objects, together with a composition operation satisfying the following properties:

- for all $f\colon X \to Y$ and $g\colon Y \to Z$ morphisms in $\mathbf{C}$, the composition $g \circ f\colon X \to Z$ of f and g is a morphism in $\mathbf{C}$;
- for all $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ morphisms in $\mathbf{C}$, it holds that $h \circ (g \circ f) = (h \circ g) \circ f$;
- for every object X in $\mathbf{C}$, there exists a morphism $\mathbf{1}_X\colon X \to X$, called the identity, such that, for all morphisms $f\colon X \to Y$ and $g\colon Y \to X$, it holds that $\mathbf{1}_Y \circ f = f$ and $g \circ \mathbf{1}_X = g$.

A category $\mathbf{C}$ is called *small* if the classes $\mathbf{Ob}_{\mathbf{C}}$ and $\mathbf{Hom}_{\mathbf{C}}$ are sets. The symbol $\mathbf{Hom}_{\mathbf{C}}(X, Y)$ denotes the subclass of $\mathbf{Hom}_{\mathbf{C}}$ of all morphisms $X \to Y$ in $\mathbf{C}$.

If a category has only the identity morphisms is called *discrete*.

We now present two examples of categories. In both of them, the composition of morphisms is the standard composition of functions, and the identity is the standard identity function.

Set: Objects are sets, and morphisms are the functions between them.

Ab: Objects are abelian groups, and morphisms are the group homomorphisms between them.

If $\mathbf{C}$ is a category, the opposite category $\mathbf{C}^{\mathbf{op}}$ has the same objects as $\mathbf{C}$ and opposite morphisms. This means that a morphism $X \to Y$ in $\mathbf{C}^{\mathbf{op}}$ is given by a morphism $Y \to X$ in $\mathbf{C}$, and the composition of morphisms $g \circ f$ in $\mathbf{C}$ is the composition $f \circ g$ in $\mathbf{C}^{\mathbf{op}}$.

To handle categories in less abstract terms, we can ask them to be small. However, this is a strong requirement. For example, **Set** is not small. The following definition provides a less strict condition.

Definition 1.2. A category $\mathbf{C}$ is said to be *locally small* if $\mathbf{Hom}_{\mathbf{C}}(X, Y)$ is a set, for all objects X and Y of $\mathbf{C}$.

Set is locally small. All the considered categories in this thesis are locally small.

Considering morphisms, one may need to generalise the notion of invertibility in categorical language. The two-sided invertibility has a direct correspondence to the idea of isomorphism. The one-sided invertibility corresponds to the notion of being a section or a retraction. Typically, they are more restrictive than being an epimorphism or a monomorphism. As a consequence, every isomorphism is, in particular, an epimorphism and a monomorphism, but the converse is not valid in general.

Definition 1.3. A morphism $f\colon X \to Y$ in a category $\mathbf{C}$

- is an *isomorphism* if it admits a two-sided inverse, which is a morphism $g\colon Y \to X$ in $\mathbf{C}$ such that $g \circ f = \mathbf{1}_X$ and $f \circ g = \mathbf{1}_Y$;

- has a *section* if it admits a right-inverse, which is a morphism $s\colon Y \to X$ such that $f \circ s = \mathbf{1}_Y$;
- has a *retraction* if it admits a left-inverse, which is a morphism $r\colon Y \to X$ such that $r \circ f = \mathbf{1}_X$.

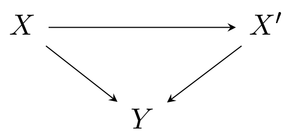
Having a section implies the right-cancellative property, and having a retraction implies the left-cancellative property. The notion of monomorphism and epimorphism is more general than the notion of a section and a retraction.

Definition 1.4. Let $f\colon X \to Y$ be a morphism in a category $\mathbf{C}$.

- f is an *epimorphism* if it is right-cancellative, i.e. if it holds that $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$, for all morphisms $g_1, g_2\colon Y \to Z$ in $\mathbf{C}$;
- f is a *monomorphism* if it is left-cancellative, i.e. if it holds that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$, for all morphisms $g_1, g_2\colon Z \to X$ in $\mathbf{C}$. Monomorphisms are denoted by the symbol $\rightarrowtail$.

The notion of isomorphism leads to an equivalence relation on objects, where two objects are equivalent if there is an isomorphism between them.

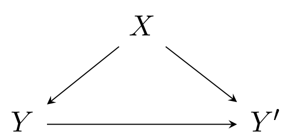
Let Y be an object in the category $\mathbf{C}$. Consider the category $\mathbf{C} \downarrow Y$, whose objects are morphisms $X \to Y$ in $\mathbf{C}$. A morphism in $\mathbf{C} \downarrow Y$ between $X \to Y$ and $X' \to Y$ is a morphism $X \to X'$ in $\mathbf{C}$ such that the following diagram commutes:



In this category, two objects $f\colon X \to Y$ and $g\colon X' \to Y$ are isomorphic if there exists an isomorphism $h\colon X \to X'$ in $\mathbf{C}$ such that $f = g \circ h$. For two such isomorphic objects, one is an isomorphism (resp. epimorphism, monomorphism) if and only if so is the other one.

Definition 1.5. A *subobject* of an object Y in a category $\mathbf{C}$ is the equivalence class of a monomorphism $f\colon X \to Y$ with respect to the isomorphism relation in $\mathbf{C} \downarrow Y$.

Dually, fix an object X in the category $\mathbf{C}$, and consider the category $X \uparrow \mathbf{C}$, whose objects are morphisms $X \to Y$ in $\mathbf{C}$. A morphism in $X \uparrow \mathbf{C}$ between $X \to Y$ and $X \to Y'$ is a morphism $Y \to Y'$ such that the following diagram commutes:



In this category, two objects $f\colon X \to Y$ and $g\colon X \to Y'$ are isomorphic if there exists an isomorphism $h\colon Y \to Y'$ in $\mathbf{C}$ such that $g = h \circ f$. For two such isomorphic objects, one is an isomorphism (resp. epimorphism, monomorphism) if and only if so is the other.

Definition 1.6. A *quotient* of an object X in a category $\mathbf{C}$ is the equivalence class of an epimorphism $f\colon X \to Y$ with respect to the isomorphism relation in $X \uparrow \mathbf{C}$.

We can now describe morphisms between categories, which are called functors.

Definition 1.7. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *functor* $F\colon \mathbf{C} \to \mathbf{D}$ associates

(i) with each object X in $\mathbf{C}$ an object $F(X)$ in $\mathbf{D}$;

(ii) with each morphism $f\colon X \to Y$ in $\mathbf{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathbf{D}$, such that:

- for every object X in $\mathbf{C}$, $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$;
- for all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathbf{C}$, it holds that $F(g \circ f) = F(g)\, F(f)$;

We use the symbol $F\colon \mathbf{Hom}_{\mathbf{C}}(X, Y) \to \mathbf{Hom}_{\mathbf{C}}(F(X), F(Y))$ to denote the function between the sets of morphisms. We say that the objects in $\mathbf{D}$ are indexed, or parametrised, by $\mathbf{C}$. We refer to $\mathbf{C}$ as the *source category*, and to $\mathbf{D}$ as the *target category*.

As first examples of a functor, we define the hom-functors. Recall our global assumption that $\mathbf{C}$ is locally small.

Definition 1.8. Let $\mathbf{C}$ be a category and X an object in $\mathbf{C}$.

- The functor $\mathbf{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \mathbf{Set}$ associates
  (i) to each object Y the set of morphisms $\mathbf{Hom}_{\mathbf{C}}(X, Y)$;
  (ii) to each morphism $f\colon Y \to Z$ the morphism $\mathbf{Hom}_{\mathbf{C}}(X, f) : \mathbf{Hom}_{\mathbf{C}}(X, Y) \to \mathbf{Hom}_{\mathbf{C}}(X, Z)$ that maps g in $\mathbf{Hom}_{\mathbf{C}}(X, Y)$ to $f \circ g$.
- The functor $\mathbf{Hom}_{\mathbf{C}}(-, X) : \mathbf{C}^{\mathbf{op}} \to \mathbf{Set}$ associates
  (i) to each object Y the set of morphisms $\mathbf{Hom}_{\mathbf{C}}(Y, X)$;
  (ii) to each morphism $f\colon Y \to Z$ the morphism $\mathbf{Hom}_{\mathbf{C}}(f, X) : \mathbf{Hom}_{\mathbf{C}}(Z, X) \to \mathbf{Hom}_{\mathbf{C}}(Y, X)$ that maps g in $\mathbf{Hom}_{\mathbf{C}}(Z, X)$ to $g \circ f$.

One may need a functor to satisfy specific properties. In particular, we are interested in two properties: being full and being faithful.

Definition 1.9. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is

- *full* if, for every pair of objects X, Y in $\mathbf{C}$, the function $F\colon \mathbf{Hom_C}(X,Y) \to \mathbf{Hom_C}(F(X), F(Y))$ is surjective;
- *faithful* if, for every pair of objects X, Y in $\mathbf{C}$, the function $F\colon \mathbf{Hom_C}(X,Y) \to \mathbf{Hom_C}(F(X), F(Y))$ is injective.

A functor which is both full and faithful is called *fully faithful*.

Example 1.10. A *poset* is a category $\mathbf{I}$ such that:

(i) for every pair of objects X, Y in $\mathbf{I}$ there is at most one morphism from $X \to Y$;

(ii) if there is a morphism from $X \to Y$ and a morphism from $Y \to X$ in $\mathbf{I}$, then $X = Y$.

This definition is the categorical interpretation of the definition of a partially ordered set, which is a set endowed with a reflexive, antisymmetric, and transitive binary relation. Two types of posets are of interest in our work: the poset of nonnegative real numbers, denoted by $[0,\infty)$, and the poset of the first n natural numbers, denoted by $[n]$. To describe a functor $[n] \to [0,\infty)$ it is enough to specify a sequence of numbers $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ of $[0,\infty)$. Such a functor is fully faithful if the sequence is strictly increasing. From now on, we identify functors $[n] \to [0,\infty)$ with such sequences, and denote them by $[n] \subset [0,\infty)$. ■

Functors themselves form a category, where morphisms are given by natural transformations.

Definition 1.11. Let F and G be functors between the categories $\mathbf{C}$ and $\mathbf{D}$. A *natural transformation* $\eta\colon F \to G$ assigns to every object X in $\mathbf{C}$ a morphism $\eta_X\colon F(X) \to G(X)$ in $\mathbf{D}$ (called the component of η at X) such that for any morphism $f\colon X \to Y$ in $\mathbf{C}$, it holds that $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Let $\eta\colon F \to G$ and $\mu\colon G \to H$ be natural transformations between the functors $F, G, H\colon \mathbf{C} \to \mathbf{D}$. The composition $\mu\eta$ is defined to be a natural transformation from F to H given by the collection of all morphisms $\mu_X\eta_X\colon F(X) \to H(X)$, for every object X in $\mathbf{C}$. This composition fulfils all the requirements needed to define a category. We use the symbol $\mathbf{fun}(\mathbf{C},\mathbf{D})$ to denote the category whose objects are functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are natural transformations with the composition defined above. We say that the objects in $\mathbf{fun}(\mathbf{C},\mathbf{D})$ are functors *indexed* by $\mathbf{C}$ with *values* in $\mathbf{D}$. Even if $\mathbf{C}$ and $\mathbf{D}$ are locally small, this category may fail to be locally small. To assure local smallness of $\mathbf{fun}(\mathbf{C},\mathbf{D})$, we need to assume $\mathbf{C}$ to be small. In our work, we study the functor categories where $\mathbf{C}$ is a poset, specifically $[0,\infty)$. Denote by $\mathbf{fun}([0,\infty), \mathbf{D})$ the category of objects in $\mathbf{D}$ parametrised by $[0,\infty)$. Since $[0,\infty)$ is small, if $\mathbf{D}$ is locally small, so is $\mathbf{fun}([0,\infty), \mathbf{D})$.

Limits and colimits

Limits and colimits are important constructions in category theory. Constructions such as pushouts, pullbacks, and initial and terminal objects are examples of limits and colimits.

Definition 1.12. Let $F\colon \mathbf{J} \to \mathbf{C}$ be a functor. A *cone* to F is an object N of $\mathbf{C}$ together with a family $\psi_I\colon N \to F(I)$ of morphisms indexed by the objects I of $\mathbf{J}$, such that, for every morphism $f\colon I \to J$ of $\mathbf{J}$, it holds that $\psi_J = F(f) \circ \psi_I$.

Dually, a *cocone* of F is an object N of $\mathbf{C}$ together with a family $\psi_I\colon F(I) \to N$ of morphisms indexed by the objects I of $\mathbf{J}$, such that for every morphism $f\colon I \to J$ in $\mathbf{J}$, it holds that $\psi_I = \psi_J \circ F(f)$.

We can now define limits and colimits.

Definition 1.13. Let $F\colon \mathbf{J} \to \mathbf{C}$ be a functor. A *limit* of F is a terminal cone (L, φ) to F, i.e. a cone such that for any other cone (N, ψ) to F there exists a unique morphism $u\colon N \to L$ such that $\varphi_I \circ u = \psi_I$ for all I in $\mathbf{J}$. Dually, a *colimit* of F is an initial cocone (L, φ) of F, i.e. a cocone such that for any other cocone (N, ψ) of F there exists a unique morphism $u\colon L \to N$ such that $u \circ \varphi_I = \psi_I$ for all I in $\mathbf{J}$

In the previous definition, a (co)limit is called small (resp. finite) whenever $\mathbf{J}$ is so. A category $\mathbf{C}$ is said to have *small (co)limits* if all small (co)limits exist in $\mathbf{C}$. In the notation of the (co)limit, if the morphisms of the (co)cone are clear, we avoid to specify them and simply write the (co)limit as L.

Remark 1.14. In general, it is not always true that (co)limits exist in a category $\mathbf{C}$. However, by (co)cone universality, for any two (co)limits, there is a unique isomorphism between them.

In a functor category $\mathbf{fun}(\mathbf{C}, \mathbf{D})$, if $\mathbf{D}$ has (co)limits, then so does $\mathbf{fun}(\mathbf{C}, \mathbf{D})$, and they are computed objectwise.

As examples of limits, we present three constructions.

Terminal object. Let $\mathbf{J}$ be the empty category. For any category $\mathbf{C}$, there is a unique functor index by $\mathbf{J}$ with values in $\mathbf{C}$. Its limit is called the *terminal object* of $\mathbf{C}$. An object $*$ is terminal if and only if there is a unique morphism from X to $*$ for every object X in $\mathbf{C}$.

Product. Let $\mathbf{J}$ be a discrete category and $\mathbf{C}$ a category. A functor indexed by $\mathbf{J}$ with values in $\mathbf{C}$ is a collection of objects $\{X^j\}_j$ of $\mathbf{C}$. The limit of such a functor is called *product* and denoted by $\prod_j X^j$.

Pullback. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be morphisms in $\mathbf{C}$. We can think of the commutative square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

as the cone of f and g. If such a cone is terminal, the commutative square is called a *pullback square*. We also refer to the limit of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ as the *pullback of f and g*. A morphism given by the universal property of the pullback is called *mediating morphism of pullback*.

As examples of colimits, we present four constructions.

Initial object. Let $\mathbf{J}$ be the empty category. Recall the unique functor indexed by $\mathbf{J}$ with values in $\mathbf{C}$. Its colimit is called the *initial* object of $\mathbf{C}$. An object $\emptyset$ is initial if and only if there is a unique morphism from $\emptyset$ to X for every object X in $\mathbf{C}$.

Coproduct. Let $\mathbf{J}$ be a discrete category and $\mathbf{C}$ a category. Recall that a functor indexed by $\mathbf{J}$ is a collection of objects $\{X^j\}_j$ of $\mathbf{C}$. The colimit of such a functor is called *coproduct* and denoted by $\coprod_j X^j$.

Pushout. Let $f\colon Z \to X$ and $g\colon Z \to Y$ be morphisms in $\mathbf{C}$. We can think of the commutative square

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & Q \end{array}$$

as the cocone of f and g. If such a cocone is initial, the commutative square is called a *pushout square*. We also refer to the colimit of a diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ as the *pushout of f and g*. A morphism given by the universal property of the pushout is called *mediating morphism of pushout*.

Directed colimit. Let $\mathbf{J}$ be a *directed set*, i.e. a set endowed with a preorder in which any finite subset has an upper bound, and $\mathbf{C}$ a category. A functor whose source category is a directed set $\mathbf{J}$ and taget category is $\mathbf{C}$ is called a *directed system* over $\mathbf{J}$ in $\mathbf{C}$. The colimit of a directed system in $\mathbf{C}$ is called *directed colimit*.

Fix a category $\mathbf{C}$ and a directed set $\mathbf{J}$. A directed system in $\mathbf{C}$ is also denoted by $\{X^i\}$. A morphism of directed systems $f\colon \{X^i\} \to \{Y^i\}$ in $\mathbf{C}$ over $\mathbf{J}$ consists of morphisms $\{f^i\colon X^i \to Y^i\}_{i\in\mathbf{J}}$ such that the following diagram commutes for all $j \leqslant i \in \mathbf{J}$:

$$\begin{array}{ccc} X^j & \xrightarrow{X^{j<i}} & X^i \\ {\scriptstyle f^j}\downarrow & & \downarrow{\scriptstyle f^i} \\ Y^j & \xrightarrow{Y^{j<i}} & Y^i \end{array}$$

It is then possible to define the category $\mathbf{DS_C}$ of directed systems over $\mathbf{J}$ in $\mathbf{C}$. The objects of the category are directed systems over $\mathbf{J}$ in $\mathbf{C}$, and the morphisms are the above-defined morphisms between them. The composition is defined pointwise and satisfies the axioms of a category.

Proposition 1.15. *If the category* $\mathbf{C}$ *admits all directed colimits, then there exists a functor* $\operatorname{dcolim}\colon \mathbf{DS_C} \to \mathbf{C}$*, sending each directed systems to its directed colimit, and each morphism of directed systems to the unique morphism given by the universal property of colimit.*

Proof. We need to prove that the assignment dcolim satisfies the axioms of a functor. Consider the identity morphism $\mathbf{1}_{\{X^i\}}\colon \{X^i\} \to \{X^i\}$ and the colimit $(\operatorname{dcolim} X, \varphi_i)$ of a directed system $\{X^i\}$. Then we have that $\operatorname{dcolim} \mathbf{1}_{\{X^i\}} \circ \varphi_i = \varphi_i$ and $\operatorname{dcolim} \mathbf{1}_{\operatorname{dcolim} X} \circ \varphi_i = \varphi_i$. By uniqueness of the colimit, it follows that $\operatorname{dcolim} \mathbf{1}_{\{X^i\}} = \mathbf{1}_{\operatorname{dcolim} X}$. Consider now three directed systems $\{X^i\}$, $\{Y^i\}$ and $\{Z^i\}$, with morphisms $f\colon \{X^i\} \to \{Y^i\}$ and $g\colon \{Y^i\} \to \{Z^i\}$, such that the composition $g \circ f$ is defined. We show that $\operatorname{dcolim}\left(g^i \circ f^i\right) = \operatorname{dcolim}\left(g^i\right) \operatorname{dcolim}\left(f^i\right)$. Let $(\operatorname{dcolim} X, \varphi)$, $(\operatorname{dcolim} Y, \psi)$ and $(\operatorname{dcolim} Z, \zeta)$ be the colimits of $\{X^i\}$, $\{Y^i\}$ and $\{Z^i\}$ respectively. Then

$$\begin{aligned}\operatorname{dcolim}\left(g^i \circ f^i\right) \circ \varphi_i = \zeta_i \circ \left(g^i \circ f^i\right) = \left(\zeta_i \circ g^i\right) \circ f^i = \left(\operatorname{dcolim} g^i \circ \psi_i\right) \circ f^i \\ = \operatorname{dcolim} g^i \circ \left(\psi_i \circ f^i\right) = \operatorname{dcolim} g^i \operatorname{dcolim} f^i\end{aligned}$$

where at each step we are using the commutativity of the cocones. This proves the claim. □

By Remark 1.14, for any two initial (resp. terminal) objects there is a unique isomorphism between them. In general, the initial and terminal objects do not coincide. When they do, they are called the *zero object*, denoted by 0. As for initial and terminal objects, for any two zero objects there is a unique isomorphism between them.

Consider the following commutative diagram:

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y_1 & \longrightarrow & Y_2 \end{array}$$

Suppose it is a pullback square. A morphism property $\mathcal{P}$ is said to be *stable under pullback* if whenever g has $\mathcal{P}$ then also f has it. Dually, suppose the diagram is a pushout square. A morphism property $\mathcal{P}$ is said to be *stable under pushout* if whenever f has property $\mathcal{P}$ then also g has it. The following proposition provides an example of stability. For a proof, see [32].

Proposition 1.16. *Isomorphisms are stable under pushout.*

The following proposition is known as the *pasting law* for pullbacks (resp. pushouts), and relate the pullback (resp. pushout) of two juxtaposed diagrams. For a proof, see Chapter 5 of [4].

Proposition 1.17. *Let the following diagram be commutative in a category* $\mathbf{C}$*:*

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \end{array}$$

If the right square is a pullback, then the outer square is a pullback if and only if the left square is a pullback.
If the left square is a pushout, then the outer square is a pushout if and only if the right square is a pushout.

The notion of a kernel of a morphism in category theory generalises the idea of kernels in algebra. There are categories for which kernels do not exist.

Definition 1.18. Let $\mathbf{C}$ be a category with initial object $\emptyset$. The kernel of a morphism $f\colon X \to Y$ is an isomorphism equivalence class in $\mathbf{C} \downarrow X$ represented by an object $\ker f \to X$ that fits in a pullback square:

$$\begin{array}{ccc} \ker f & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

The existence of kernels depends on the existence of pullbacks and the initial object.

Another important notion is the notion of a cokernel of a morphism, which generalises the idea of cokernels in algebra. There are categories for which cokernels do not exist.

Definition 1.19. Let $\mathbf{C}$ be a category with terminal object $*$. The *cokernel* of a morphism $f\colon X \to Y$ is an isomorphism equivalence class in $Y \uparrow \mathbf{C}$ represented by an object $Y \to \operatorname{coker} f$ that fits in a pushout square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & \operatorname{coker} f \end{array}$$

The existence of cokernels depends on the existence of pushouts and the terminal object.

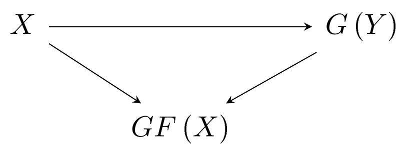
Adjoint functors

The following relation between two functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ has proved to be important.

Definition 1.20. $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ are called *adjoint functors* (F *left-adjoint* to G and G *right-adjoint* to F), if there exists a natural isomorphism between the hom-functors

$$\mathbf{Hom_D}(F(-), -) \cong \mathbf{Hom_C}(-, G(-))$$

Explicitly, a right-adjoint to $F\colon \mathbf{C} \to \mathbf{D}$ is a functor $G\colon \mathbf{D} \to \mathbf{C}$ together with a natural transformation $\mathbf{1} \to GF$ such that for any morphism $X \to G(Y)$ in $\mathbf{C}$ there is a unique morphism $F(X) \to Y$ in $\mathbf{D}$ such that the following diagram commutes



As an example, colim: $\mathbf{fun}(\mathbf{C^{op}}, \mathbf{D}) \to \mathbf{D}$ is the right-adjoint of the constant functor $\Delta\colon \mathbf{D} \to \mathbf{fun}(\mathbf{C^{op}}, \mathbf{D})$, which sends every object of $\mathbf{D}$ to the diagram functor constant on this object.

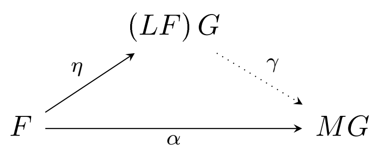
Left Kan extension

In this section, we discuss the left Kan extension, which is the most fundamental notion in category theory. As MacLane said, 'The notion of Kan extensions subsumes all the other fundamental concepts of category theory'[28].

Let $G\colon \mathbf{C} \to \mathbf{C}'$ be a functor, and $\mathbf{D}$ a category. The induced functor between functor categories $G^*\colon \mathbf{fun}(\mathbf{C}', \mathbf{D}) \to \mathbf{fun}(\mathbf{C}, \mathbf{D})$ sends each functor $H\colon \mathbf{C}' \to \mathbf{D}$ to the composition HG.

Definition 1.21. The left adjoint of G^* is called the *left Kan extension* along G.

Explicitly, the left Kan extension of a funtor $F\colon \mathbf{C} \to \mathbf{D}$ with respect to a functor $G\colon \mathbf{C} \to \mathbf{C}'$ is a functor $LF\colon \mathbf{C}' \to \mathbf{D}$ together with a natural transformation $\eta\colon F \to (LF)G$ such that η satisfies the following property: for every functor $M\colon \mathbf{C}' \to \mathbf{D}$ and natural transformation $\alpha\colon F \to MG$, there is a unique natural transformation $\gamma\colon LF \to M$ that fits in the following commutative diagram:



1.2 Abelian categories

To be able to use algebraic techniques, we need to put additional structure on a category.

Definition 1.22. A category $\mathbf{C}$ is *pre-additive* if, for each pair of objects X and Y in $\mathbf{C}$, $\mathbf{Hom}_{\mathbf{C}}(X, Y)$ is an additive abelian group, and the composition of morphisms is bilinear with respect to this addition.

Definition 1.23. An *abelian category* $\mathbf{C}$ is a pre-additive category satisfying the following conditions:

(i) $\mathbf{C}$ has the zero object;

(ii) $\mathbf{C}$ has all finite products and coproduts;

(iii) Every morphism in $\mathbf{C}$ has a kernel and cokernel;

(iv) Every monomorphism is a kernel, and every epimorphism is a cokernel.

One of the examples we have presented before, $\mathbf{Ab}$, is in particular an example of an abelian category.

Remark 1.24. For every abelian category $\mathbf{A}$, the following properties hold:

1. Finite coproducts coincide with finite products and are called *direct sums*. This allows us to talk about the decomposition of objects;
2. If $\mathbf{J}$ is an arbitrary small category, then $\mathbf{fun}(\mathbf{J}, \mathbf{A})$ is abelian. This property provides a condition for functor categories to be abelian, and it is extremely useful for us. In fact, all the functor categories of our interest fulfil this condition and thus are abelian;
3. In $\mathbf{A}$, all finite limits and colimits exist.

In particular, the first property leads to the following definition:

Definition 1.25. An object in an abelian category $\mathbf{A}$ is *indecomposable* if it is not isomorphic to a direct sum of at least two non-zero objects of $\mathbf{A}$.

The following result shows that isomorphisms, epimorphisms and monomorphisms are stable under pushouts and pullbacks. For the proof, see Section 13 of [32].

Proposition 1.26. *Consider the following diagram in an abelian category* $\mathbf{A}$*:*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{g'} & W \end{array} \tag{1.2.1}$$

If (1.2.1) is a pushout and g is a monomorphism (resp. an epimorphism), then g' is a monomorphism (resp. an epimorphism). Moreover, the induced morphism $\operatorname{coker}(g) \to \operatorname{coker}(g')$ *is an isomorphism.*
If (1.2.1) is a pullback and g' is an epimorphism (resp. a monomorphism), then g is an epimorphism (resp, a monomorphism). Moreover, the induced morphism $\ker(g) \to \ker(g')$ *is an isomorphism.*

Definition 1.27. In an abelian category $\mathbf{A}$, let $\cdots \xrightarrow{f_{j+1}} X_j \xrightarrow{f_j} X_{j-1} \xrightarrow{f_{j-1}} \cdots$ be a possibly infinite sequence of morphisms $f_j\colon X_j \to X_{j-1}$. Such a sequence is said to be *exact* if $\operatorname{im}(f_{j+1}) = \ker(f_j)$, for all j. An exact sequence of the form $0 \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \longrightarrow 0$ is called *short.*

In a short exact sequence, the morphism f_2 is a monomorphism, and the morphism f_1 is an epimorphism.

In the following proposition, we define and characterise the split exact sequences. For a proof, see Chapter 1 of [29].

Proposition 1.28. *In an abelian category $\mathbf{A}$, the short exact sequence*

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$

is said to split *if any of the following equivalent conditions holds:*

- *There exists a* section *of p, i.e. a monomorphism $s\colon Z \to Y$ such that $p \circ s = \mathbf{1}_Z$;*
- *There exists a* retraction *of i, i.e. an epimorphism $r\colon Y \to X$ such that $r \circ i = \mathbf{1}_X$;*
- *There exists an isomorphism $f\colon Y \to X \oplus Z$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbf{1}} & & \\ 0 & \longrightarrow & X & \xrightarrow{i'} & X \oplus Z & \xrightarrow{p'} & Z & \longrightarrow & 0 \end{array}$$

  where i' is the inclusion into the first factor, and p' is the projection onto the second factor.

The two following results are fundamental lemmas about exact sequences. For their proofs, we refer to Chapter 5 of [40].

Lemma 1.29 (Five lemma)**.** *In an abelian category $\mathbf{A}$, consider the commutative diagram with exact rows*

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_5 \end{array}$$

- *If f_2 and f_4 are epimorphisms and f_5 is a monomorphism, then f_3 is an epimorphism.*
- *If f_2 and f_4 are monomorphisms and f_1 is an epimorphism, then f_3 is a monomorphism.*
- *If f_1, f_2, f_4, and f_5 are isomorphisms, then f_3 is an isomorphism.*

Lemma 1.30 (Short five lemma)**.** *In an abelian category* $\mathbf{A}$*, consider the commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0 \end{array}$$

if f_1 and f_3 are monomorphisms, then f_2 is a monomorphism; if f_1 and f_3 are epimorphisms, then f_2 is an epimorphism.

The following two propositions state some properties of kernels and cokernels. For the proof, we refer to [41].

Proposition 1.31. *Given the following commutative solid diagram with exact rows in an abelian category* $\mathbf{A}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ & & {\scriptstyle f}\,\vdots & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \end{array}$$

there exists a unique morphism $f\colon X_1 \to Y_1$, making the whole diagram commute.

Proposition 1.32. *Given the following commutative solid diagram with exact rows in an abelian category* $\mathbf{A}$

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \vdots\,{\scriptstyle f} & & \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0 \end{array}$$

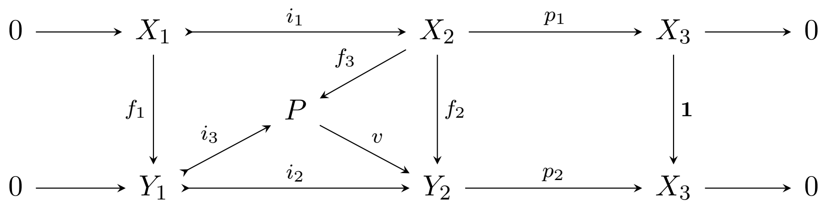
there exists a unique morphism $f\colon X_3 \to Y_3$, making the whole diagram commute.

To conclude the section, we present a proposition that combines exact sequences and pushouts.

Proposition 1.33. *In an abelian category* $\mathbf{A}$*, if the following diagram is commutative with exact rows, then the left square is a pushout:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{p_1} & X_3 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \mathbf{1}} & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{i_2} & Y_2 & \xrightarrow{p_2} & X_3 & \longrightarrow & 0 \end{array}$$

Proof. Since the rows are exact sequences, i_1 and i_2 are monomorphisms and p_1 and p_2 are epimorphisms. To prove the claim, we consider the pushout P of i_1 and f_1 and show it is isomorphic to Y_2. The pushout P fits into the following commutative diagram:



where v is the mediating morphism induced of the pushout P, and i_3 is a monomorphism by Proposition 1.26. Moreover, since p_1 is an epimorphism and the diagram commutes, also $p_2 \circ v$ is an epimorphism. We can expand the diagram as follows, ensuring the exactness of rows and the commutativity:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{p_1} & X_3 & \longrightarrow & 0 \\
 & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \mathbf{1}} & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{i_3} & P & \xrightarrow{p_2 \circ v} & X_3 & \longrightarrow & 0 \\
 & & {\scriptstyle \mathbf{1}}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \mathbf{1}} & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{i_2} & Y_2 & \xrightarrow{p_2} & X_3 & \longrightarrow & 0
\end{array}$$

By Lemma 1.30, v is an isomorphism, proving the claim. □

We now present a detailed example of an abelian category, the category of vector spaces. We are especially interested in vector spaces because they are the building blocks of the other categories we are working with.

Vector spaces

Let us fix a field $\mathtt{k}$. For the rest of the work, all vector spaces are defined over $\mathtt{k}$. The category of $\mathbf{Vect}_{\mathtt{k}}$ is the category whose objects are vector spaces, and whose morphisms are linear transformations. The composition is the standard composition of linear transformations, and it satisfies the axioms of a category. Moreover, $\mathbf{Vect}_{\mathtt{k}}$ is abelian. The zero object is the trivial vector space. Note that $\mathbf{Vect}_{\mathtt{k}}$ is not small, but it is locally small [27]. The indecomposable objects in $\mathbf{Vect}_{\mathtt{k}}$ admit a straightforward characterisation: an object V in $\mathbf{Vect}_{\mathtt{k}}$ is indecomposable if and only if $\dim V = 1$ or $\dim V = 0$. From Remark 1.24, it follows that pushouts and pullbacks exist in $\mathbf{Vect}_{\mathtt{k}}$. Moreover, they admit an explicit description. We describe the pullback construction first.

Let W_1, W_2 and U be vector spaces in $\mathbf{Vect}_{\mathtt{k}}$. Let $g_1 \colon W_1 \to U$ and $g_2 \colon W_2 \to U$ be two morphisms in $\mathbf{Vect}_{\mathtt{k}}$. Define $P := \{(w_1, w_2) \mid g_1(w_1) = g_2(w_2)\} \subseteq W_1 \oplus W_2$. The pullback of g_1 and g_2 is (P, π_1, π_2), where π_i, $i = 1, 2$, are the projections onto the first and the second component respectively. As a pullback diagram:

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & W_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ W_1 & \xrightarrow{g_1} & U \end{array} \tag{1.2.2}$$

We describe now the pushout construction. Let W_1, W_2 and V be vector spaces in $\mathbf{Vect_k}$. Let $f_1 \colon V \to W_1$ and $f_2 \colon V \to W_2$ be two morphisms in $\mathbf{Vect_k}$. Define $Q := {W_1 \oplus W_2}/{\sim}$, where $\sim$ is the equivalence relation given by $(f_1(v), 0) \sim (0, f_2(v))$ for all $v \in V$. The pushout of f_1 and f_2 is (Q, ι_1, ι_2), where ι_i, $i = 1, 2$, are the inclusions into the first and the second component respectively. As a pushout diagram:

$$\begin{array}{ccc} V & \xrightarrow{f_2} & W_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ W_1 & \xrightarrow{\iota_1} & Q \end{array} \tag{1.2.3}$$

In $\mathbf{Vect_k}$, the pullback and the pushout of a square are linked by the following proposition. Let the following be a commutative diagram of vector spaces:

$$\begin{array}{ccc} V & \xrightarrow{f_2} & W_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ W_1 & \xrightarrow{g_1} & U \end{array} \tag{1.2.4}$$

Lemma 1.34. *Let P and Q be respectively the pullback and the pushout of g_1, g_2 and f_1, f_2 in diagram (1.2.4), with f and g the unique mediating morphisms of P and Q:*

Then $f \colon V \to P$ is an epimorphism if and only if $g \colon Q \to U$ is a monomorphism.

Proof. Suppose that f is an epimorphism and g is not a monomorphism. Then there exists $w = (w_1, w_2) \neq 0$ in Q, such that $g(w_1, w_2) = 0$. Since the diagram with the mediating morphism g commutes, we have $g_1(w_1) = -g_2(w_2) = g_2(-w_2)$. Thus, $(w_1, -w_2) \in P$. Since f is an epimorphism, there exists $v \in V$ such that $f(v) = (w_1, -w_2)$. Since the diagram with the mediating morphism f commutes, $w_1 = f_1(v)$ and $-w_2 = f_2(v)$. In other words, $(w_1, 0) \sim (0, -w_2)$, and $(w_1, w_2) = 0$ in Q, which is a contradiction. Thus g is a monomorphism.

Suppose now that g is a monomorphism and f is not an epimorphism. Then there exists $(w_1, w_2) \in P$ such that $(w_1, w_2) \notin \mathrm{im}(f)$. Since $(w_1, w_2) \in P$, $g_1(w_1) = g_2(w_2) \in U$. Since $(w_1, w_2) \notin \mathrm{im}(f)$, $(w_1, 0) \nsim (0, w_2)$. It follows that the equivalence classes

$(w_1, 0)$ and $(0, w_2)$ are two distinct elements in Q such that $g(w_1, 0) = g(0, w_2)$, which is a contradiction. Thus, f is an epimorphism. □

Often, we do not deal with single vector space, but rather with a family of vector spaces, indexed by natural numbers. We define then the category $\mathbf{GVect}_{\mathtt{k}}$ of graded vectors spaces. An object in $\mathbf{GVect}_{\mathtt{k}}$ is a $\mathtt{k}$-vector space graded over the natural numbers $\mathbb{N}$, which is a sequence of vector spaces $V = \{V_h\}_{h\in\mathbb{N}}$. A morphism between graded vector spaces is a sequence of linear transformations $\{V_h \to W_h\}_{h\in\mathbb{N}}$, and the composition is the composition of linear transformations degreewise. Such a composition fulfils the axioms of a category. Moreover, $\mathbf{GVect}_{\mathtt{k}}$ is abelian. The zero object is the graded vector space which is zero in all degrees. A graded vector space V is said to be *concentrated in degree* $n \in \mathbb{N}$ if $V_h = 0$ for all $h \neq n$. There is a fully faithful functor $\mathbf{Vect}_{\mathtt{k}} \to \mathbf{GVect}_{\mathtt{k}}$ that maps vector spaces to graded vector spaces concentrated in degree 0.

We introduce now a construction over graded vector spaces that will be useful in the decomposition of chain complexes in Section 3.2.

Construction 1.35. Let us consider a graded vector space V. Define SV to be the graded vector space given by:

$$SV_h := \begin{cases} 0 & \text{if } h = 0 \\ V_{h-1} & \text{if } h \geqslant 1 \end{cases}$$

Such graded vector space is called the *suspension* of V. ■

Let $V = \mathtt{k}$. The h-fold suspension of V is a graded vector space concentrated in degree h. Such a graded vector space is called h*-sphere*, or simply *sphere*, and denoted by S^h. Explicitly, the h-sphere S^h is depicted in the following diagram:

$$\begin{array}{ccccccccc} \cdots & & h{+}1 & & h & & h{-}1 & & \cdots \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

1.3 Chain complexes

A (non-negatively graded) chain complex is a sequence of linear transformations $X = \{\partial_{h+1} \colon X_{h+1} \to X_h\}_{h\in\mathbb{N}}$, called *differentials*, such that $\partial_h \circ \partial_{h+1} = 0$ for every degree $h \in \mathbb{N}$. In the notation of the differentials, we often ignore their indexes and simply denote them by ∂, or ∂^X to indicate which complex we are considering. A morphism of chain complexes $f \colon X \to Y$ is a sequence of linear transformations $\{f_h \colon X_h \to Y_h\}_{h\in\mathbb{N}}$ such that $f_h \circ \partial^X = \partial^Y \circ f_{h+1}$ for every $h \in \mathbb{N}$. We also refer to a morphism of chain complexes as a *chain map*. We denote by $\mathbf{Ch}$ the category whose objects are chain complexes and whose morphisms are chain maps. A classical result shows that **Ch** is abelian [47]. The zero object is given by the chain complex zero in all degrees. The direct sum of chain complexes is given degreewise. In particular, let $Y = \left(Y_h, \partial_h^Y\right)$ and $Z = \left(Z_h, \partial_h^Z\right)$ be two chain complexes, their direct sum is the chain complex $X = \left(X_h, \partial_h^X\right)$ with $X_h = Y_h \oplus Z_h$ and $\partial_h^X = \partial_h^Y \oplus \partial_h^Z$. Note that **Ch** is not small, but it is locally small.

There is a fully faithful functor $\mathbf{GVect}_\Bbbk \to \mathbf{Ch}$ that maps graded vector spaces to chain complexes with all trivial differentials. Recall the fully faithful functor $\mathbf{Vect}_\Bbbk \to \mathbf{GVect}_\Bbbk$ that we defined in Section 1.2. The composition of these two functors is a fully faithful functor $\mathbf{Vect}_\Bbbk \to \mathbf{Ch}$ that maps vector spaces to chain complexes nontrivial only in degree 0.

Let V be a graded vector space, and consider the chain complex associated to it by the functor $\mathbf{GVect}_\Bbbk \to \mathbf{Ch}$. The same symbol V is also used to denote such a chain complex, i.e. the chain complex $\{0\colon V_{h+1} \to V_h\}_{h\in\mathbb{N}}$ with trivial differentials.

We present three functors that map from chain complexes to graded vector spaces. The following graded vector spaces are called respectively the *cycles* and the *boundaries* of a chain complex X:

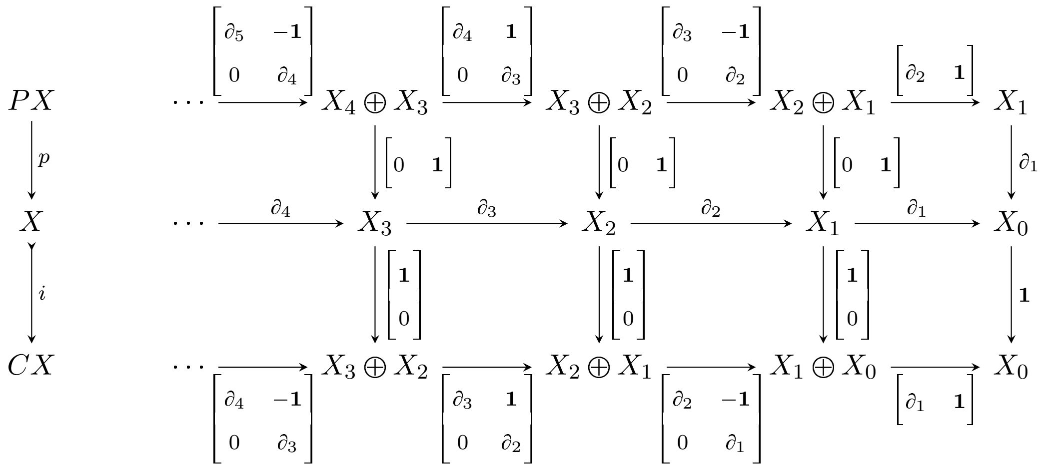
$$Z_hX := \begin{cases} X_0 & \text{if } h = 0 \\ \ker\left(\partial_h\colon X_h \to X_{h-1}\right) & \text{if } h \geqslant 1 \end{cases}, \quad B_hX := \operatorname{im}\left(\partial_{h+1}\colon X_{h+1} \to X_h\right)$$

Since $\partial_h \circ \partial_{h+1} = 0$, the space of h-th boundaries B_hX is a vector subspace of the h-th cycles Z_hX. The quotient Z_hX/B_hX is called the h-th *homology* of X and is denoted by H_hX. We write ZX, BX and HX to denote the non-negatively graded vector spaces $\{Z_hX\}_{h\in\mathbb{N}}$, $\{B_hX\}_{h\in\mathbb{N}}$, and $\{H_hX\}_{h\in\mathbb{N}}$. A chain map $f\colon X \to Y$ maps boundaries (resp. cycles) in X to boundaries (resp. cycles) in Y. As a consequence, the assignments $X \mapsto BX$, $X \mapsto ZX$ and $X \mapsto HX$ define three functors $B, Z, H\colon \mathbf{Ch} \to \mathbf{GVect}_\Bbbk$. Given a chain map $f\colon X \to Y$, the induced map in homology is denoted by $Hf\colon HX \to HY$.

Homology admits a more general description. In fact, homology is a functor $H_h\colon \mathbf{Ch} \to \mathbf{Ab}$, which associates to a chain complex X in **Ch** the vector space $H_hX = \operatorname{coker}\left(\operatorname{im}\partial_{h+1} \to \ker\partial_h\right)$, for all $h \in \mathbb{N}$. As shown in Chapter 1 in [47], homology is functorial and it preserves direct sums. See [29] for extended discussion and proof.

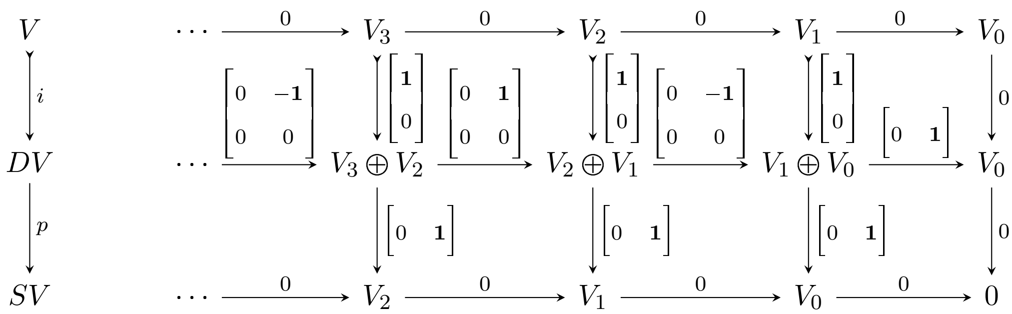
Definition 1.36. A chain complex X is *bounded* if there exists $m \in \mathbb{N}$ such that $X_h \cong 0$ for all $h > m$, and *degreewise finite dimensional* if for all $h \in \mathbb{N}$, X_h is a finite-dimensional vector space.

Construction 1.37. [Cone and Path] Let X be a chain complex in **Ch**. We define two chain complexes CX and PX over X, together with the morphisms $i\colon X \rightarrowtail CX$ and $p\colon PX \to X$, as follows:



The assignements $X \mapsto CX$ and $X \mapsto PX$ define two functors $\mathbf{Ch} \to \mathbf{Ch}$. The chain complex CX is called *cone* of X and the chain complex PX is called the *path* of X. Note that $HPX = HCX = 0$. ■

As an example of the cone construction, we present the cone of graded vector spaces. Let V be a graded vector space. The cone of V is denoted by DV, and it is explicitly described in the following diagram:



The morphisms $i\colon V \rightarrowtail CV$ and $p\colon PSV \to SV$ defined in Construction 1.37 coincide with respectively $i\colon V \rightarrowtail DV$ and $p\colon DV \to SV$. Moreover, $i\colon V \rightarrowtail DV$ is a monomorphism and $p\colon DV \to SV$ is an epimorphism, and they form an exact sequence:

$$0 \longrightarrow V \xrightarrow{\ i\ } DV \xrightarrow{\ p\ } SV \longrightarrow 0$$

Note that $H_h DV = 0$ for all $h \in \mathbb{N}$, $H_h SV = H_{h-1} V = V_{h-1}$ for all $h \geqslant 1$, and $H_0 SV = 0$.

If $V = \mathtt{k}$, the h-fold suspension of DV is a chain complex trivial in all degrees except for degrees $h, h-1$, with identity as boundary map in degree h. It is called the h-*disk*, or simply *disk*, D^h. Explicitly:

$$\begin{array}{ccccccccccc} \scriptstyle\cdots & & \scriptstyle h+1 & & \scriptstyle h & & \scriptstyle h-1 & & \scriptstyle h-2 & & \scriptstyle\cdots \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\ \mathbf{1}\ } & \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Note that, when $h = 0$, $D^0 \cong S^0$.

Construction 1.38. [Mapping cylinder] Given a chain map $f\colon X \to Y$ in **Ch**, we construct the mapping cylinder $MC(f)$ of f as follows:

$$\cdots \xrightarrow{\begin{bmatrix} \partial_3^X & \mathbf{1} & 0 \\ 0 & \partial_2^X & 0 \\ 0 & f_2 & \partial_3^Y \end{bmatrix}} X_2 \oplus X_1 \oplus Y_2 \xrightarrow{\begin{bmatrix} \partial_2^X & -\mathbf{1} & 0 \\ 0 & \partial_1^X & 0 \\ 0 & f_1 & \partial_2^Y \end{bmatrix}} X_1 \oplus X_0 \oplus Y_1 \xrightarrow{\begin{bmatrix} \partial_1^X & \mathbf{1} & 0 \\ 0 & f_0 & \partial_1^Y \end{bmatrix}} X_0 \oplus Y_0$$

Define the morphism $p\colon MC(f) \to Y$ to be the projection on the last component. In particular, p is an epimorphism and induces an isomorphism in homology. Define the morphisms $i\colon X \to MC(f)$ as the inclusion into the first component. In particular, i is a monomorphism. ■

A proof of the following theorem can be found in Chapter 5 of [40].

Theorem 1.39. *Let the following be a commutative diagram with exact rows in* **Ch***:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' & \longrightarrow & 0
\end{array}$$

Then there is a commutative diagram in homology with exact rows:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_h X & \xrightarrow{Hi} & H_h Y & \xrightarrow{Hp} & H_h Z & \xrightarrow{\delta_h} & H_{h-1} X & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle Hf_h} & & \downarrow{\scriptstyle Hf'_h} & & \downarrow{\scriptstyle Hf''_h} & & \downarrow{\scriptstyle Hf_{h-1}} & & \\
\cdots & \longrightarrow & H_h X' & \xrightarrow{Hi'} & H_h Y' & \xrightarrow{Hp'} & H_h Z' & \xrightarrow{\delta'_h} & H_{h-1} X' & \longrightarrow & \cdots
\end{array}$$

The morphisms δ and δ' are called connecting morphisms*, and for every cycle $z \in Z_h X$ and $z' \in Z_h X'$, they are given by*

$$\begin{aligned} \delta_h [z] &\mapsto \left[i_{h-1}^{-1} \circ \partial_h^X \circ p_h^{-1}(z) \right] \\ \delta'_h [z'] &\mapsto \left[i'^{-1}_{h-1} \circ \partial_h^{X'} \circ p'^{-1}_h (z') \right] \end{aligned} \tag{1.3.1}$$

We conclude these preliminaries describing two particular types of chain complexes. Let h be a natural number. A chain complex X is *concentrated in degree* h if $X_l = 0$ for all $l \neq h$. Note that any such chain complex is isomorphic to a direct sum $\oplus S^h$ of a certain number of h-dimensional spheres [25]. A chain complex is concentrated in *one degree* if, for some natural number h', it is concentrated in degree h'.

A chain complex X is concentrated in degrees $\{h, h+1\}$ if $X_l = 0$ for all $l \notin \{h, h+1\}$. Note that any such chain complex is a direct sum $(\oplus S^h) \oplus (\oplus S^{h+1}) \oplus (\oplus D^{h+1})$ of certain number of complexes S^h, S^{h+1} and D^{h+1} [25]. A chain complex is concentrated in *two consecutive degrees* if, for some natural number h', it is concentrated in degrees $\{h', h'+1\}$. Note that any complex concentrated in one degree is also concentrated in two degrees, and the zero chain complex is concentrated in degree h and in degrees $h, h+1$, for any h in $\mathbb{N}$.

Let X be a chain complex concentrated in degree h, Y a chain complex concentrated in degrees $\{h, h+1\}$ and Z a chain complex concentrated in degrees $\{h-1, h\}$, with $h > 0$. Then the following morphisms are bijections:

$$\mathbf{Hom}_{\mathbf{ch}}(X, Y) \to \mathbf{Hom}_{\mathbf{vect}_{\Bbbk}}(X_h, Y_h) \qquad f \mapsto f_h$$
$$\mathbf{Hom}_{\mathbf{ch}}(Z, Y) \to \mathbf{Hom}_{\mathbf{vect}_{\Bbbk}}(Z_h, Y_h) \qquad f \mapsto f_h$$
$$\mathbf{Hom}_{\mathbf{ch}}(Z, X) \to \mathbf{Hom}_{\mathbf{vect}_{\Bbbk}}(Z_h, X_h) \qquad f \mapsto f_h$$

Chapter 2

Parametrised objects

In this chapter, we are going to discuss functors indexed by the posets $[0, \infty)$ and the poset of the first n natural numbers, denoted by $[n]$. A functor indexed by $[n]$ with values in a category $\mathbf{C}$ is a sequence of n composable morphisms $X^0 \to X^1 \to \cdots \to X^n$ in $\mathbf{C}$. If $\mathbf{C}$ is the category of finite-dimensional vector spaces, then specifying such a functor requires only a finite amount of information. A functor indexed by $[0, \infty)$ consists of an infinite collection of morphisms $X^{s<t} \colon X^s \to X^t$ indexed by any pair of numbers $s < t$ in $[0, \infty)$, such that the following diagram commutes for any $s < t < r$ in $[0, \infty)$:

$$X^s \xrightarrow{X^{s<t}} X^t \xrightarrow{X^{t<r}} X^r, \qquad X^{s<r} = X^{t<r} \circ X^{s<t}$$

Describing a functor parametrised by $[0, \infty)$ requires an infinite amount of information. Such functors can be very complicated, in general. For data analysis purposes, however, we do not need to deal with intricacies of arbitrary functors indexed by $[0, \infty)$. Since our work is driven by the goal of analysing data through computable descriptors, we focus on functors indexed by finite subposets of $[0, \infty)$. Note that any finite subposet of $[0, \infty)$ is of the form $[n]$, for various n. Thus, the category of functors indexed by $[0, \infty)$ provides a convenient universe containing functors indexed by $[n]$ for all n. After the general discussion about parametrised objects in a category $\mathbf{C}$, we describe two categories of parametrised objects: the category of parametrised vector spaces and the category of parametrised chain complexes.

We then characterise the compact objects in the categories of interest. The compact parametrised chain complexes are called *tame*, and they form the central category of our work. The study of compact objects is motivated in data analysis since data are finite, but it also has a theoretical motivation: in the abelian categories of compact objects, every object admits a unique decomposition into indecomposables.

Finally, we motivate the study of tame parametrised chain complexes showing that they include three important classes of objects. Such classes are the category of parametrised vector spaces, known in the literature as persistent modules and central objects of study in persistent homology, the category of commutative ladders, introduced in the contest of TDA in [9, 17], and the category of zigzag, known in the literature as zigzag modules. The zigzag modules are proven to be engaging in TDA, but their signatures are not yet efficiently computable [11, 12, 13].

2.1 From finite posets to $[0, \infty)$

In this section, we discuss a standard way of extending a functor indexed by $[n]$ to a functor indexed by $[0, \infty)$ along an inclusion $[n] \subset [0, \infty)$, called the left Kan extension (Definition 1.21). In general, the existence of the left Kan extension depends on the properties of the category in which the considered functors take values. In spite of this, in the case of a functor indexed by $[n]$ the left Kan extension along an inclusion $[n] \subset [0, \infty)$ always exists. It is important to note that no condition is required on the target category $\mathbf{C}$ for existing the left Kan extension in this setting. The category $\mathbf{C}$ does not need to admit colimits, nor to have any other specific property: the left Kan extension along $[n] \subset [0, \infty)$ exists and is explicitly described by virtue of the total order of $[0, \infty)$. Therefore, for an inclusion $[n] \subset [0, \infty)$, instead of defining its left Kan extension using universal property, we give its explicit description.

Let $X = \left(X^0 \to X^1 \to \cdots \to X^n\right)$ be a functor indexed by a finite subposet $t_0 < t_1 < \cdots < t_n$ of $[0, \infty)$. For t in $[0, \infty)$ set

$$LX^t := X^{\max\{i \mid t_i \leqslant t\}}$$

Note that if $s < t$, then $\max\{i \mid t_i \leqslant s\} \leqslant \max\{i \mid t_i \leqslant t\}$, and hence we can define $LX^{s<t} \colon LX^s \to LX^t$ to be the composition of the morphism $X^{\max\{i \mid t_i \leqslant s\}} \to \cdots \to X^{\max\{i \mid t_i \leqslant t\}}$. We also refer to LX as the extension of the n composable morphisms $X = \left(X^0 \to X^1 \to \cdots \to X^n\right)$ along the sequence $t_0 < t_1 < \cdots < t_n$. The left Kan extension $L \colon \mathbf{fun}\left([n], \mathbf{C}\right) \to \mathbf{fun}\left([0, \infty), \mathbf{C}\right)$ is left adjoint to the restriction functor $\mathbf{fun}\left([0, \infty), \mathbf{C}\right) \to \mathbf{fun}\left([n], \mathbf{C}\right)$ that maps i to t_i along the inclusion $[n] \subset [0, \infty)$. This means that to describe a natural transformation $\eta \colon LX \to Y$ from such a Kan extension to any other functor $Y \colon [0, \infty) \to \mathbf{C}$ it is enough to specify a sequence of morphisms $\left\{f^{t_i} \colon LX^{t_i} \to Y^{t_i}\right\}_{i=0,\dots,n}$ for which the following diagram commutes, for all $i = 0, \dots, n$:

$$\begin{array}{ccc} LX^{t_i} & \xrightarrow{LX^{t_i<t_{i+1}}} & LX^{t_{i+1}} \\ {\scriptstyle f^{t_i}}\downarrow & & \downarrow{\scriptstyle f^{t_{i+1}}} \\ Y^{t_i} & \xrightarrow{Y^{t_i<t_{i+1}}} & Y^{t_{i+1}} \end{array}$$

In particular $L \colon \mathbf{fun}\left([n], \mathbf{C}\right) \to \mathbf{fun}\left([0, \infty), \mathbf{C}\right)$ is fully faithful.

Remark 2.1. Let LX be the left Kan extension of an object X in $\mathbf{fun}([n], \mathbf{C})$ along the sequence $0 = t_0 < \cdots < t_n$. LX satisfies the following property: $LX^{s<t}$ may fail to be an isomorphism only if there is an i such that $s < t_i \leqslant t$. In particular, the restrictions of the transitions of X to the left-closed and right-open intervals $[t_0, t_1)$, ..., $[t_{n-1}, t_n), [t_n, \infty)$ are isomorphisms.

2.2 Discretisation

In this section, we describe the full subcategory of $\mathbf{fun}([0, \infty), \mathbf{C})$ given by the union of the images of the left Kan extensions along all inclusions of the form $[n] \subset [0, \infty)$, for all n. These are the functors we focus on in our work. A fundamental property of such functors is described in Remark 2.1.

Definition 2.2. An object X in $\mathbf{fun}([0, \infty), \mathbf{C})$ is called *discretisable* if there exists a sequence $0 = t_0 < t_1 < \cdots < t_n$ in $[0, \infty)$ such that each transition morphism $X^{s<t} \colon X^s \to X^t$ may fail to be an isomorphism only when there exists $i \in \{1, \ldots, n\}$ such that $s < t_i \leqslant t$. The sequence $0 = t_0 < t_1 < \cdots < t_n$ is said to *discretise* X.

Discretising sequences have the following properties. Given a sequence $0 = t_0 < t_1 < \cdots < t_n$ in $[0, \infty)$ discretising X, every finite refinement is also a discretising sequence for X. Moreover, any finite collection of discretisable objects in $\mathbf{fun}([0, \infty), \mathbf{C})$ admits a common discretising sequence. Such a sequence is given by the union of elements of the discretising sequences, which is a refinement of all of them.

Theorem 2.3. *Let $Y \colon [0, \infty) \to \mathbf{C}$ be a functor. Then Y is isomorphic to a left Kan extension LX along some inclusion $[n] \subset [0, \infty)$ if and only if it is discretisable.*

Proof. If Y is isomorphic to a left Kan extension LX along some inclusion $[n] \subset [0, \infty)$, by Remark 2.1 it is discretisable.

If Y is discretisable it admits a discretising sequence $0 = t_0 < t_1 < \cdots < t_n$. We identify such a sequence with an inclusion $[n] \subset [0, \infty)$ (Example 1.10). Let X be the restriction of Y along this inclusion. Explicitly, $X \colon [n] \to \mathbf{C}$ is $X^i = Y^{t_i}$, $i = 0, \ldots, n$ with transition morphisms $X^{i<i+1} := Y^{t_i<t_{i+1}}$. Let $LX \to Y$ be the adjoint to $\mathbf{1} \colon X \to X$. This is an isomorphism. □

2.3 Relevant categories

In this section, we describe the functor category $\mathbf{fun}([0, \infty), \mathbf{Ch})$, whose objects are functors indexed by $[0, \infty)$ taking values in the category of chain complexes. We introduce the discussion presenting the category of parametrised vector spaces.

Parametrised vector spaces

Recall that a parametrised vector space is an object in the category $\mathbf{fun}\left([0,\infty), \mathbf{Vect}_{\mathtt{k}}\right)$. Such a category is abelian by Remark 1.24. Its zero object is the functor mapping to the zero vector space at every step. We start describing indecomposable objects in $\mathbf{fun}\left([0,\infty), \mathbf{Vect}_{\mathtt{k}}\right)$. Choose b in $[0,\infty)$. Consider the inclusion $0 \to b$. Let $\mathtt{k}\colon [0] \to \mathbf{Vect}_{\mathtt{k}}$ be a functor sending 0 to the one dimensional vector space. We use the symbol $\mathbb{I}_{[b,\infty)}$ to denote the left Kan extension of this functor along the inclusion $b\colon [0] \subset [0,\infty)$. Explicitly:

$$\mathbb{I}^t_{[b,\infty)} = \begin{cases} \mathtt{k} & \text{if } t \geqslant b \\ 0 & \text{otherwise} \end{cases}$$

Choose two elements $b < d$ in $[0,\infty)$. The morphism $b < d$ in $[0,\infty)$ induces the inclusion $[1] \subset [0,\infty)$. Consider the functor $\mathtt{k} \to 0\colon [1] \to \mathbf{Vect}_{\mathtt{k}}$, sending 0 to $\mathtt{k}$ and 1 to 0. We use the symbol $\mathbb{I}_{[b,d)}$ to denote the left Kan extension this functor along this inclusion. Explicitly:

$$\mathbb{I}^t_{[b,d)} \cong \begin{cases} \mathtt{k} & \text{for } t \in [b,d) \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{I}^{s\leqslant t}_{[b,d)} \cong \begin{cases} \mathbf{1} & \text{if } s \leqslant t \in [b,d) \\ 0 & \text{otherwise} \end{cases}$$

We call the functor $\mathbb{I}_{[b,\infty)}$ and $\mathbb{I}_{[b,d)}$ *interval vector spaces.* Interval vector spaces are indecomposables. Indeed, consider, with a little abuse of notation, $b < d \leqslant \infty$ and $e < f \leqslant \infty$. The morphisms between $\mathbb{I}_{[b,d)}$ and $\mathbb{I}_{[e,f)}$ are either zero or one dimensional:

$$\mathbf{Hom}_{\mathbf{fun}([0,\infty),\mathbf{Vect}_{\mathtt{k}})}\left(\mathbb{I}_{[b,d)}, \mathbb{I}_{[e,f)}\right) \text{ is } \begin{cases} \text{1-dimensional} & \text{if } e \leqslant b \text{ and } f \leqslant d \\ \text{0-dimensional} & \text{otherwise} \end{cases}$$

It follows that the endomorphism ring of an interval vector space is local, and thus the object is indecomposable.

Parametrised chain complexes

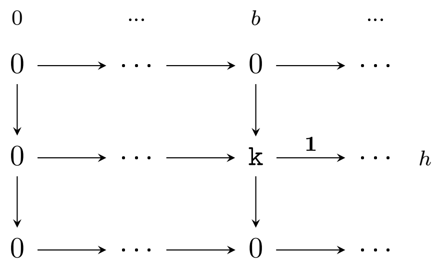
A parametrised chain complex is an object in the functor category $\mathbf{fun}\left([0,\infty), \mathbf{Ch}\right)$. The category $\mathbf{fun}\left([0,\infty), \mathbf{Ch}\right)$ is abelian by Remark 1.24. Its zero object is the functor whose values are the chain complex trivial in all degrees. The category $\mathbf{fun}\left([0,\infty), \mathbf{Ch}\right)$ is of wild representation type (see Section 2.5 - Commutative ladders). Thus, it is not possible to list all its indecomposables. Among its indecomposables, three families of functors play an essential role in our work. The elements of these families are called *interval spheres.*

The first two families are parametrised by b in $[0,\infty)$ and $h \in \mathbb{N}$. Consider the induced functor $b\colon [0] \to [0,\infty)$ and the inclusion $b\colon [0] \subset [0,\infty)$.

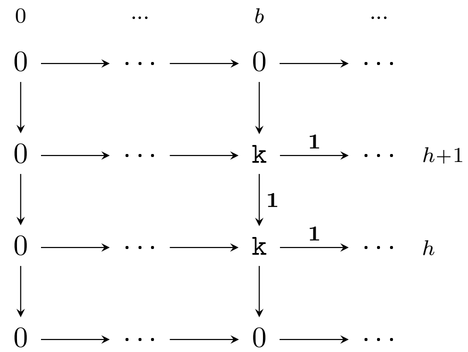
- $I^h[b,\infty)$ denotes the left Kan extension of the functor $S^h\colon [0] \to \mathbf{Ch}$ along the inclusion $b\colon [0] \subset [0,\infty)$, assigning $0 \mapsto S^h$;
- $I^h[b,b)$ denotes the left Kan extension of the functor $D^{h+1}\colon [0] \to \mathbf{Ch}$ along the inclusion $b\colon [0] \subset [0,\infty)$, assigning $0 \mapsto D^{h+1}$.

Explicitly, these functors can be depicted as follows:

$LS^h = I^h[b,\infty)$



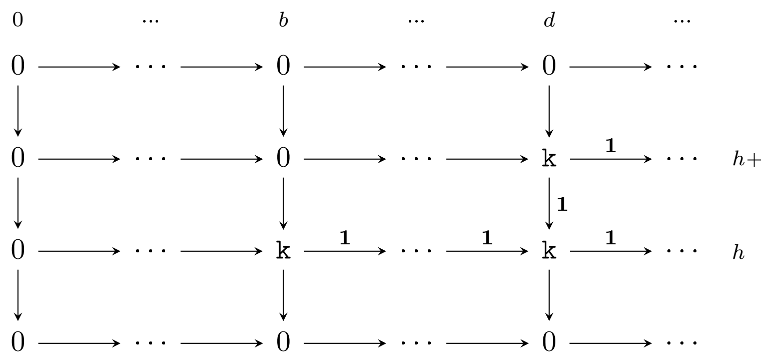
$LD^{h+1} = I^h[b,b)$



The third family is indexed by pairs of elements in $[0,\infty)$ and $h \in \mathbb{N}$. Consider the functor $b < d\colon [1] \to [0,\infty)$, and the induced inclusion $b < d\colon [1] \subset [0,\infty)$.

- $I^h[b,d)$ denotes the left Kan extension of the functor $S^h \to D^{h+1}\colon [1] \to \mathbf{Ch}$, sending 0 to S^h and 1 to D^{h+1}, along the inclusion $b < d\colon [1] \subset [0,\infty)$.

Such a functor can be graphically described as follows:



Homology provides a relation between the objects $I^h[b,d)$ and the objects $\mathbb{I}_{[b,d)}$:

- The parametrised vector space $H_h I^h[b,\infty)$ is isomorphic to $\mathbb{I}_{[b,\infty)}$;
- The parametrised vector space $H_h I^h[b,b)$ is isomorphic to 0;
- The parametrised vector space $H_h I^h[b,d)$ is isomorphic to $\mathbb{I}_{[b,d)}$.

Consider $\mathbf{Hom}_{\mathbf{fun}([0,\infty),\mathbf{Ch})}\left(I^h[b,d), I^{h'}[e,f)\right)$, where, with a little abuse of notation, d and f can be infinite. We have:

$$\mathbf{Hom}_{\mathbf{fun}([0,\infty),\mathbf{Ch})}\left(I^h[b,d), I^{h'}[e,f)\right) \text{ is } \begin{cases} \text{1-dimensional} & \text{if } h = h' \text{ and } e \leqslant b \leqslant f \leqslant d \\ \text{1-dimensional} & \text{if } h = h' + 1 \text{ and } f \leqslant b \\ \text{0-dimensional} & \text{otherwise} \end{cases}$$

Thus, the endomorphism ring of every interval sphere is local, and hence interval spheres are indecomposable. Here, there are some examples of the morphisms of interval spheres. Between $I^h[b,d)$ and $I^h[b,b)$ there is a morphism $i\colon I^h[b,d) \to I^h[b,b)$, induced by the following diagrams:

	$I^h[b,d)$	$I^h[b,b)$	$I^h[b,\infty)$
$\begin{array}{c} I^h[b,d) \\ \downarrow{\scriptstyle i} \\ I^h[b,b) \end{array}$	$\begin{array}{ccccc} 0 & \longrightarrow & S^h & \xhookrightarrow{i} & D^{h+1} \\ \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathbf{1}} \\ 0 & \longrightarrow & D^{h+1} & \xrightarrow{\mathbf{1}} & D^{h+1} \end{array}$	$\begin{array}{ccc} 0 & \longrightarrow & D^{h+1} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{1}} \\ 0 & \longrightarrow & D^{h+1} \end{array}$	$\begin{array}{ccc} 0 & \longrightarrow & S^h \\ \downarrow & & \downarrow{\scriptstyle i} \\ 0 & \longrightarrow & D^{h+1} \end{array}$

2.4 Compactness

In this section, we discuss the notion of compactness in the categories of interest. For a reference of compactness in category theory, see [1].

Definition 2.4. Let $\mathbf{C}$ be a category where all directed colimits exist. An object X in $\mathbf{C}$ is called *compact* if $\mathbf{Hom}_{\mathbf{C}}(X,-)$ commutes with all directed colimits. Explicitly, X is compact if the natural map $\operatorname{colim}\mathbf{Hom}_{\mathbf{C}}(X,Y) \to \mathbf{Hom}_{\mathbf{C}}(X,\operatorname{colim}Y)$ is an isomorphism for every functor Y indexed by a directed set with values in $\mathbf{C}$.

More in details, an object X in a category $\mathbf{C}$ is compact if and only if for each colimit $\left(\operatorname{colim}D, d^i\right)$ of a directed system $\{D^i\}$ over $\mathbf{J}$ in $\mathbf{C}$, and each morphism $f\colon X \to \operatorname{colim}D$ there exists $i \in \mathbf{J}$ such that

- f factorises through d^i, i.e. $f = d^i \circ g$, for some $g\colon X \to D^i$;
- the factorisation is essentially unique, in the sense that if $f = d^i \circ g = d^i \circ g'$, then $D^{i\leqslant j} \circ g = D^{i\leqslant j} \circ g'$, for some $i \leqslant j \in \mathbf{J}$.

The notion of compactness is crucial, as we will see in Remark 2.17. Moreover, it allows the following definition.

Definition 2.5. Let $\mathbf{C}$ be a category. The functor $X\colon [0,\infty) \to \mathbf{C}$ is called *tame* if it is discretisable and X^t compact for each t in $[0,\infty)$.

Remark 2.6. The full subcategory of an abelian category consisting of all its compact objects is abelian.

We now characterise the compact objects in $\mathbf{Vect_k}$, $\mathbf{Ch}$, $\mathbf{fun}\left([0,\infty), \mathbf{Vect_k}\right)$ and $\mathbf{fun}\left([0,\infty), \mathbf{Ch}\right)$.

Compact vector spaces

We begin characterising compact vector spaces.

Proposition 2.7. *An object V in $\mathbf{Vect_k}$ is compact if and only if its dimension is finite.*

Proof. Suppose V is compact. Since every vector space is the directed colimit of all its finite dimensional linear subspaces, the linear transformation $\mathbf{1}_V : V \to V$ factorises through the inclusion into a finite dimensional linear subspace. Hence, V is finite dimensional.

Suppose V is finite dimensional. Let $\mathcal{B}$ be a basis of V in $\mathbf{Vect_k}$. Let $\left(\operatorname{colim} W, w^i\right)$ be the colimit of a directed system $\{W^i\}$ over $\mathbf{J}$, and $f : V \to \operatorname{colim} W$ a linear transformation. For each element $v \in \mathcal{B}$ there exists $i_v \in \mathbf{J}$ such that $f(v)$ lies in the image of w^{i_v}. Since V is finite dimensional and $\mathbf{J}$ is directed, there exists an upper bound $i \in \mathbf{J}$ of all i_v, $v \in \mathcal{B}$. Thus, $f(V) \subset w^i\left(W^i\right)$ and f factorises through w^i. The essential uniqueness follows from the fact that, for vector spaces, two elements are equal in the directed colimit if and only if they eventually become equal. □

We denote by $\mathbf{vect_k}$ the full subcategory of $\mathbf{Vect_k}$ whose objects are compact vector spaces and whose morphisms are linear maps between them. By Remark 2.6, $\mathbf{vect_k}$ is abelian. The category $\mathbf{vect_k}$ has some nice properties. For example, in $\mathbf{vect_k}$, every section is a monomorphism, and every retraction is an epimorphism. Moreover, every monomorphism has a retract, and every epimorphism has a section.

Compact chain complexes

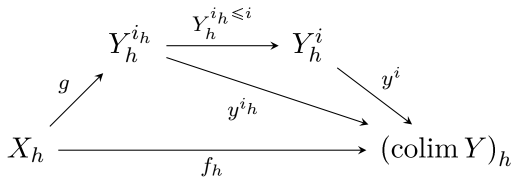
To study compact chain complexes, recall Definition 1.36 of bounded and degreewise finite dimensional chain complexes.

Remark 2.8. In $\mathbf{Ch}$, the functor colim is exact.

Proposition 2.9. *An object X in $\mathbf{Ch}$ is compact if and only if it is degreewise finite dimensional and bounded.*

Proof. Suppose X compact. By Proposition 2.7, if there exists a degree in which X is not finite dimensional, then X is not compact, which is a contradiction. Assume then that X is degreewise finite dimensional. From a direct computation, it follows that X can be written as the directed colimit of the family of chain complexes $\{\overline{X}^i\}_{i\in\mathbb{N}}$ where $\overline{X}^i_h = X_h$ for all $h \leqslant i \in \mathbb{N}$, and zero otherwise, for all $i \in \mathbb{N}$. Then, by compactness, the chain map $\mathbf{1}_X : X \to X$ factorises through X^i for some $i \in \mathbb{N}$. Since X^i is bounded, the claim follows.

Suppose X is degreewise finite dimensional and bounded. Let $\left(\operatorname{colim} Y, y^i\right)$ be the colimit of a directed system $\{Y^i\}$ over $\mathbf{J}$ in $\mathbf{Ch}$, and $f\colon X \to \operatorname{colim} Y$ a chain map. By Proposition 1.15 and Remark 2.8, $\operatorname{colim} Y$ can be computed degreewise, using the differentials as morphisms of directed systems. Then, by Proposition 2.7, it follows that, in each degree h, there exists an essentially unique factorisation of f_h for some $i_h \in \mathbf{J}$. Since X is bounded, there exists $m \in \mathbb{N}$ such that $X_h = 0$ for all $h > m$. Since $\mathbf{J}$ directed, there exists an upper bound $i \in \mathbf{J}$ of all i_h, $0 \leqslant h \leqslant m$. We claim that then the chain map f factorises through Y^i. To see it, it is enough to check the commutativity degreewise, for all $0 \leqslant h \leqslant m$, as shown in the following diagram:



By Proposition 2.7, $f_h = y^{i_h} \circ g$, and by definition of cocone $y^{i_h} = y^i \circ Y_h^{i_h \leqslant i}$. Thus, $f_h = y^i \circ Y_h^{i_h \leqslant i} \circ g$ and f_h factorises through Y_h^i. The essential uniqueness follows from Proposition 2.7, since it is can be verified degreewise. ☐

We denote by $\mathbf{ch}$ the full subcategory of compact objects of $\mathbf{Ch}$. By Remark 2.6, $\mathbf{ch}$ is abelian.

Remark 2.10. In algebra, ascending and descending families of subobjects are proved to be interesting. One can define the ascending and the descending chain condition on subobjects, and these notions are related to being Artinian and Noetherian. We then can study the ascending chain condition and the descending chain condition on compact chain complexes. Since they are finitely dimensional and bounded, they satisfy both conditions. For a general discussion about these properties, see Chapter 6 of [3].

Compact parametrised vector spaces

To explicitly characterise compact objects in $\mathbf{fun}\left([0,\infty), \mathbf{Vect}_{\mathrm{k}}\right)$, we define a finiteness condition for parametrised vector spaces.

Definition 2.11. In $\mathbf{fun}\left([0,\infty), \mathbf{Vect}_{\mathrm{k}}\right)$, an object V is said to be *pointwise finite dimensional* if V^t is a finite dimensional vector space for all t in $[0,\infty)$.

Proposition 2.12. *An object V of $\mathbf{fun}\left([0,\infty), \mathbf{Vect}_{\mathrm{k}}\right)$ is compact if and only if it is pointwise finite dimensional and discretisable.*

Proof. Suppose V compact. By Proposition 2.7, if there exists a step in which V is not finite dimensional, then it cannot be compact. Assume then that V is pointwise finite dimensional. Let $\mathbf{J}$ be the directed poset of finite subsets of $[0,\infty)$. An element s of $\mathbf{J}$ is a set $\{s_1, s_2, \ldots, s_n\}$. Define a functor W_s as $W_s^t = V^{s_i}$ for all $t \in [s_i, s_{i+1})$, and $W_s^t = 0$ for all $t \in [0, s_1)$, if $s_1 \neq 0$. Define $f_{sr}\colon W_r \to W_s$ and $\varphi_s\colon W_s \to V$, for all $r \leqslant s \in \mathbf{J}$, as:

$$f_{sr}^t := \begin{cases} 0 & \text{if } W_r^t = 0 \\ \mathbf{1} & \text{if } W_r^t = V_s^t \\ V_s^{t' \leqslant t} & \text{if } W_r^t = V_s^{t'} \end{cases} \qquad \varphi_s^t := \begin{cases} 0 & \text{if } W_s^t = 0 \\ \mathbf{1} & \text{if } W_s^t = V^t \\ V^{t' \leqslant t} & \text{if } W_s^t = V^{t'} \end{cases}$$

A direct computation shows that (V, φ_s) is the colimit of the directed system $\{W_s\}$ over $\mathbf{J}$. Thus, by compactness, the morphism $\mathbf{1}\colon V \to V$ factorises through V_s, for some $s \in \mathbf{J}$, and V is discretisable.

Suppose V pointwise finite dimensional and discretisable, with discretising sequence $0 = t_0 < \cdots < t_n$. Let $\left(\operatorname{colim} W, w^i\right)$ be the colimit of a directed system $\{W^i\}$ over $\mathbf{J}$, and $f\colon V \to \operatorname{colim} W$ a morphism of parametrised vector spaces. By Proposition 1.15, $\operatorname{colim} W$ can be computed pointwise, using the transition morphisms $W^{t_i < t_{i+1}}$, for $i = 0, \ldots, n-1$, as morphisms of directed systems. Then, by Proposition 2.7, it follows that there exists a factorisation of f^{t_i} in each step t_i for some $j_{t_i} \in \mathbf{J}$. Since V is discretisable and $\mathbf{J}$ directed, there exists an upper bound $j \in \mathbf{J}$ of all j_{t_i}, $i = 0, \ldots, n$. Then the morphism f factorises through W^j. The argument for the factorisation and its essential uniqueness is similar to the case of Proposition 2.7 and Proposition 2.9, and we omit the details. ☐

We denote by $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\Bbbk}\right)$ the full subcategory of $\mathbf{fun}\left([0,\infty), \mathbf{Vect}_{\Bbbk}\right)$ whose objects are compact. We refer to these compact objects as *tame parametrised vector spaces*. By Remark 2.6, $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\Bbbk}\right)$ is abelian. We often depict an object in this category as follows:

$$V^0 \longrightarrow V^{t_1} \longrightarrow \cdots \longrightarrow V^{t_n}$$

We now recall the decomposition theorem for objects in $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\Bbbk}\right)$. For a proof, see [15].

Theorem 2.13. *Every object in* $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\Bbbk}\right)$ *is a direct sum of interval vector spaces. Furthermore, such a decomposition are unique up to an isomorphism.*

The previous result is a milestone in topological data analysis, providing invariants for parametrised vector spaces. Such invariants are given by the number and type of the indecomposables.

Compact parametrised chain complexes

We now define a finiteness condition for parametrised chain complexes, to explicitly characterise compact objects in $\mathbf{fun}\left([0,\infty), \mathbf{Ch}\right)$.

Definition 2.14. In $\mathbf{fun}\left([0,\infty),\mathbf{Ch}\right)$, an object X is said to be *stepwise bounded* if, for all t in $[0,\infty)$, X^t is a bounded chain complex, and *pointwise finite dimensional*, if for all t in $[0,\infty)$ and for all $h \in \mathbb{N}$, X_h^t is a finite dimensional vector space.

Proposition 2.15. *In* $\mathbf{fun}\left([0,\infty),\mathbf{Ch}\right)$*, an object X is compact if and only if it is pointwise finite dimensional, stepwise bounded and discretisable.*

Proof. Suppose X compact. By Proposition 2.12, if X is not discretisable, then it is not compact, which is a contradiction. By Proposition 2.9, if for at least one step t in $[0,\infty)$, X^t is not bounded, then it cannot be compact. By Proposition 2.7, if for at least one step t in $[0,\infty)$ and one degree h, X_h^t is not finite dimensional, then it is not compact, which is a contradiction.

Suppose X is stepwise bounded, pointwise finite dimensional, and discretisable, with discretising sequence $0 = t_0 < \cdots < t_n$. Let $\left(\operatorname{colim} Y, y^i\right)$ be the colimit of a directed system $\{Y^i\}$ over $\mathbf{J}$, and $f\colon X \to \operatorname{colim} Y$ a morphism of parametrised chain complexes. By Proposition 1.15, $\operatorname{colim} Y$ can be computed along the discretising sequence, using the transition morphisms $Y^{t_i<t_{i+1}}$, for $i = 0,\ldots,n-1$, as morphisms of directed systems. Then, by Proposition 2.9, it follows that there exists a factorisation of f^{t_i} in each step t_i for some $j_{t_i} \in \mathbf{J}$. Since X is discretisable and $\mathbf{J}$ directed, there exists an upper bound $j \in \mathbf{J}$ of all j_{t_i}, $i = 0,\ldots,n$. Then the morphism f factorises through Y^j. The commutativity follows by similar arguments to the ones used in Proposition 2.9. The essential uniqueness of the factorisation is verified stepwise, using the result of Proposition 2.7 and Proposition 2.9. We omit the details. □

We denote by $\mathbf{tame}\left([0,\infty),\mathbf{ch}\right)$ the full subcategory of $\mathbf{fun}\left([0,\infty),\mathbf{Ch}\right)$ whose objects are compact. We call the objects in $\mathbf{tame}\left([0,\infty),\mathbf{ch}\right)$ *tame parametrised chain complexes.* By Remark 2.6, $\mathbf{tame}\left([0,\infty),\mathbf{ch}\right)$ is an abelian category.

Here is a graphical representation of a tame parametrised chain complexes, discretised by $0 < t_1 < t_2 < t_3 < t_4$:

$$
\begin{array}{ccccccccccc}
0 & & t_1 & & t_2 & & t_3 & & t_4 & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle \partial_3^0} & & \downarrow{\scriptstyle \partial_3^{t_1}} & & \downarrow{\scriptstyle \partial_3^{t_2}} & & \downarrow{\scriptstyle \partial_3^{t_3}} & & \downarrow{\scriptstyle \partial_3^{t_4}} & & \\
X_2^0 & \xrightarrow{X_2^{0<t_1}} & X_2^{t_1} & \xrightarrow{X_2^{t_1<t_2}} & X_2^{t_2} & \xrightarrow{X_2^{t_2<t_3}} & X_2^{t_3} & \xrightarrow{X_2^{t_3<t_4}} & X_2^{t_4} & \xrightarrow{\mathbf{1}} & \cdots \\
\downarrow{\scriptstyle \partial_2^0} & & \downarrow{\scriptstyle \partial_2^{t_1}} & & \downarrow{\scriptstyle \partial_2^{t_2}} & & \downarrow{\scriptstyle \partial_2^{t_3}} & & \downarrow{\scriptstyle \partial_2^{t_4}} & & \\
X_1^0 & \xrightarrow{X_1^{0<t_1}} & X_1^{t_1} & \xrightarrow{X_1^{t_1<t_2}} & X_1^{t_2} & \xrightarrow{X_2^{t_2<t_3}} & X_1^{t_3} & \xrightarrow{X_2^{t_3<t_4}} & X_1^{t_4} & \xrightarrow{\mathbf{1}} & \cdots \\
\downarrow{\scriptstyle \partial_1^0} & & \downarrow{\scriptstyle \partial_1^{t_1}} & & \downarrow{\scriptstyle \partial_1^{t_2}} & & \downarrow{\scriptstyle \partial_1^{t_3}} & & \downarrow{\scriptstyle \partial_1^{t_4}} & & \\
X_0^0 & \xrightarrow{X_0^{0<t_1}} & X_0^{t_1} & \xrightarrow{X_0^{t_1<t_2}} & X_0^{t_2} & \xrightarrow{X_0^{t_2<t_3}} & X_0^{t_3} & \xrightarrow{X_0^{t_3<t_4}} & X_0^{t_4} & \xrightarrow{\mathbf{1}} & \cdots
\end{array}
$$

Krull-Schmidt categories

In the categories of compact objects we studied, it is possible to prove a generalisation of the Krull-Schmidt theorem. We refer to [24] for a proof of the Krull-Schmidt theorem for modules, and to [2, 26, 35] for its generalisation in abelian categories.

Definition 2.16. Let $\mathbf{C}$ be a preadditive category which admits finite coproducts. $\mathbf{C}$ is a *Krull-Schmidt category* if every object in $\mathbf{C}$ decomposes uniquely up to isomorphism into a finite direct sum of indecomposables.

Explicitly, let $\mathbf{C}$ be a category as in Definition 2.16. Every non-zero object X in $\mathbf{C}$ can be written as $X_1 \oplus \cdots \oplus X_l$, where X_i is indecomposable, for all $i = 1, \ldots, l$. Moreover, if $X \cong X_1 \oplus \cdots \oplus X_l$ and $X \cong X'_1 \oplus \cdots \oplus X'_{l'}$, where X_i and X'_j are indecomposable for all $i = 1, \ldots, l$ and $j = 1, \ldots, l'$, then $l = l'$ and, $X_i \cong X'_{\sigma(i)}$, for some permutation σ.

Remark 2.17. The categories $\mathbf{vect}_{\mathrm{k}}$, $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\mathrm{k}}\right)$, $\mathbf{ch}$ and $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$ are Krull-Schimdt categories. This follows from the characterisation of Krull-Schmidt categories proven in [2].

2.5 Motivational examples

Parametrised vector spaces

Recall the standard workflow of persistent homology. When applying the homology functor in degree h to a tame parametrised chain complex, we obtain a tame parametrised vector space. This is the reason why tame parametrised vector spaces are widely studied in TDA. In this subsection, we show that any tame parametrised vector space can be seen as a special type of tame parametrised chain complex.

Consider functor $E\colon \mathbf{tame}\left([0,\infty), \mathbf{vect}_{\mathrm{k}}\right) \to \mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$ that maps every tame parametrised vector space into a tame parametrised chain complex nonzero only in degree 0. Such a functor is fully faithful. We describe explicitly how it acts. Let V be an object in $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\mathrm{k}}\right)$, and f a morphism in $\mathbf{tame}\left([0,\infty), \mathbf{vect}_{\mathrm{k}}\right)$. Then

$$E(V) := X \quad \text{where} \quad \begin{aligned} X_h^t &= \begin{cases} V^t & \text{for } h = 0 \text{ and for all } t \text{ in } [0,\infty) \\ 0 & \text{otherwise} \end{cases} \\ X_h^{s<t} &= \begin{cases} V^{s<t} & \text{for } h = 0 \text{ and for all } s < t \text{ in } [0,\infty) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$E(f) := \varphi \quad \text{where} \quad \varphi_h^t = \begin{cases} f^t & \text{for } h = 0 \text{ and for all } t \text{ in } [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Commutative ladders

In this subsection, we show that commutative ladders can be seen as parameterised chain complexes. We refer to [17] for the applications of commutative ladders in topological data analysis. Commutative ladders of length $\geqslant 5$ are of wild representation type [9]. This means that, while looking for computable invariants for them, we cannot rely on their decomposition and we need to find alternative ways. In this work, we consider commutative ladders of any finite length, but with all forward maps.

Let $\mathbf{J} = \{\mathbf{0} < \mathbf{1}\}$ be a poset with only two elements. The arrow category of $\mathbf{vect}_\Bbbk$, denoted by $\mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)$, is the functor category $\mathbf{fun}\left(\mathbf{J}, \mathbf{vect}_\Bbbk\right)$. Explicitly, an object d in $\mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)$ is a linear transformation, and a morphism (f_0, f_1) in $\mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)$ is a commutative square

$$\begin{array}{ccc} V & \xrightarrow{f_1} & V' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ W & \xrightarrow{f_0} & W' \end{array}$$

Let $0 = t_0 < t_1 < \cdots < t_n$ be a sequence in $[0, \infty)$. A *tame commutative ladder* is the left Kan extension of a functor $CL \colon [n] \to \mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)$ along the inclusion $[n] \subset [0, \infty)$. We often depict a tame commutative ladder as follows:

$$\begin{array}{ccccccc} V^0 & \longrightarrow & V^{t_1} & \longrightarrow & \cdots & \longrightarrow & V^{t_n} \\ {\scriptstyle d^0}\downarrow & & \downarrow{\scriptstyle d^{t_1}} & & & & \downarrow{\scriptstyle d^{t_n}} \\ W^0 & \longrightarrow & W^{t_1} & \longrightarrow & \cdots & \longrightarrow & W^{t_n} \end{array}$$

Denote by $\mathbf{tame}\left([0, \infty), \mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)\right)$ the functor category of tame commutative ladders. Consider a functor $E \colon \mathbf{tame}\left([0, \infty), \mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)\right) \to \mathbf{tame}\left([0, \infty), \mathbf{ch}\right)$ that maps every tame commutative ladder to a tame parametrised chain complex nonzero only in degree $0, 1$. Such a functor is fully faithful. To describe how E maps, consider an object CL and a morphism $f = (f_0, f_1)$ in $\mathbf{tame}\left([0, \infty), \mathbf{Arr}\left(\mathbf{vect}_\Bbbk\right)\right)$. Then we have:

$$E(CL) := X \quad \text{where} \quad \begin{aligned} X_h^t &= \begin{cases} W^t & \text{for } h = 0 \text{ and for all } t \text{ in } [0, \infty) \\ V^t & \text{for } h = 1 \text{ and for all } t \text{ in } [0, \infty) \\ 0 & \text{otherwise} \end{cases} \\ X_h^{s<t} &= \begin{cases} W^{s<t} & \text{for } h = 0 \text{ and for all } s < t \text{ in } [0, \infty) \\ V^{s<t} & \text{for } h = 1 \text{ and for all } s < t \text{ in } [0, \infty) \\ 0 & \text{otherwise} \end{cases} \\ \partial_h^t &= \begin{cases} d^t & \text{for } h = 1 \text{ and for all } t \text{ in } [0, \infty) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$E(f) := \varphi \quad \text{where} \quad \varphi_h^t = \begin{cases} f_h^t & \text{for } h = 0,1 \text{ and for all } t \text{ in } [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Zigzags

The goal of this section is to translate zigzag modules to objects in $\mathbf{tame}([0,\infty), \mathbf{ch})$. We begin introducing the zigzag sequences and how to construct them using *stackable directed linear transformations*. We then define chain complex parametrised by $[n]$ and how to constructed them using the *concatenation* of chain complexes and directed linear transformations. The procedure mirrors the construction of zigzag sequences. Finally, we extend over $[0,\infty)$ the chain complexes parametrised over $[n]$, using the results of Section 2.2.

A *directed linear transformation* by definition is a pair (f, c) consisting of a linear transformation f between finite dimensional vector spaces over a fixed field $\mathtt{k}$, and an element c in the set $\{r, l\}$ called the *direction* of f. We are going to use the following symbols to depict a directed linear transformation depending on its direction:

- (f, l) is depicted as $\xleftarrow{f}$;
- (f, r) is depicted as $\xrightarrow{f}$.

Two directed linear transformations (f, c) and (g, c') are said to be *stackable* if one the following conditions is satisfied:

- $c = r$, $c' = r$, and the codomain of f coincide with the domain of g. The linear transformations f and g are then composable, and we can form their composition gf. We depict this case graphically as $\xrightarrow{f}\xrightarrow{g}$.
- $c = r$, $c' = l$, and the codomain of f coincide with the codomain of g. We depict this case graphically as $\xrightarrow{f}\xleftarrow{g}$.
- $c = l$, $c' = r$, and the domain of f coincide with the domain of g. We depict this case graphically as $\xleftarrow{f}\xrightarrow{g}$.
- $c = l$, $c' = l$, and the codomain of g coincide with the domain of f. The linear transformations f and g are then composable, and we can form their composition fg. We depict this case graphically as $\xleftarrow{f}\xleftarrow{g}$.

A sequence $\{(f_i, c_i)\}_{1\leqslant i\leqslant m}$ of directed linear transformations is called a *zigzag sequence* if, for every $1 \leqslant i < m$, the directed linear transformations (f_i, c_i) and (f_{i+1}, c_{i+1}) are stackable. Here are some graphical illustrations of zigzag sequences:

$$\rightarrow\leftarrow\rightarrow\leftarrow\rightarrow\cdots \qquad \leftarrow\rightarrow\rightarrow\leftarrow\rightarrow\rightarrow\cdots \qquad \rightarrow\rightarrow\leftarrow\leftarrow\rightarrow\cdots$$

Let $\{(f_i, c_i)\}_{1\leqslant i\leqslant m}$ be a zigzag sequence where every linear transformation belongs to the set $\{0 \to 0, 0 \to \mathtt{k}, \mathtt{k} \to 0, \mathbf{1}_{\mathtt{k}}\}$, and such that

(i) There exists at most one $(f_i, c_i) \in \{(0 \to \mathtt{k}, r), (\mathtt{k} \to 0, l)\}$. If there exists such a (f_i, c_i), then set $s = i$, otherwise set $s = 0$;

(ii) There exists at most one $(f_i, c_i) \in \{(\mathtt{k} \to 0, r), (0 \to \mathtt{k}, l)\}$. If there exists such a (f_i, c_i), then set $e = i$, otherwise set $e = m$;

(iii) For all $i < s$ and $i > e + 1$, $f_i \colon 0 \to 0$ and for all $s < i < e$, $f_i = \mathbf{1}_{\mathtt{k}}$.

Such a zigzag sequence is called *interval zigzag sequence* and denoted by $\{(f_i, c_i)\}^{s,e}_{1\leqslant i\leqslant m}$.

Definition 2.18. A *zigzag profile* $\mathcal{C}$ is a sequence of directions $\{c_i\}_{1\leqslant i\leqslant m}$.

Note that, while specifying a zigzag profile, we are fixing the number of directed transformations and the directions of a zigzag sequence. Once a zigzag profile $\mathcal{C} = \{c_i\}_{1\leqslant i\leqslant m}$ is fixed, we denote a zigzag sequence simply by $\{f_i\}$.

Definition 2.19. Fix a zigzag profile $\mathcal{C}$, and let $\{f_i\}$ and $\{g_i\}$ be two zigzag sequences. A morphism $\varphi\colon \{f_i\} \to \{g_i\}$ of zigzag sequences is a collection of morphisms $\{\varphi_i \colon f_i \to g_i\}_{1\leqslant i\leqslant m}$, such that, if $c_i = r$ (resp. $c_i = l$), the diagram

$$\begin{array}{ccc} & \xrightarrow{f_i} & \\ \varphi_i\big\downarrow & & \big\downarrow\varphi_{i+1} \\ & \xrightarrow[g_i]{} & \end{array} \qquad \left(\text{resp.}\ \begin{array}{ccc} & \xleftarrow{f_i} & \\ \varphi_i\big\downarrow & & \big\downarrow\varphi_{i+1} \\ & \xleftarrow[g_i]{} & \end{array}\right)$$

commutes, for $i = 1, \ldots, m - 1$.

Once we fix a zigzag profile $\mathcal{C}$, we can define the category $\mathbf{ZS}_{\mathcal{C}}$ whose objects are zigzag sequences of zigzag profile $\mathcal{C}$ and whose morphisms are given by Definition 2.19. The composition of morphisms is defined pointwise, and since the zigzag profile is fixed, it fulfils the axioms of a category. Note that the category $\mathbf{ZS}_{\mathcal{C}}$ admits the construction of the direct sum: given $\{f_i\}$ and $\{g_i\}$ two zigzag sequences in $\mathbf{ZS}_{\mathcal{C}}$, their direct sum is defined as $\{f_i\} \oplus \{g_i\} = \{(f_i, g_i)\}$. The direct sum construction allows us to decompose zigzag sequences using the following theorem. For a proof, see [11].

Theorem 2.20. *Every zigzag sequence can be written as a finite direct sum of interval zigzag sequences. Moreover, such a decomposition is unique up to isomorphisms.*

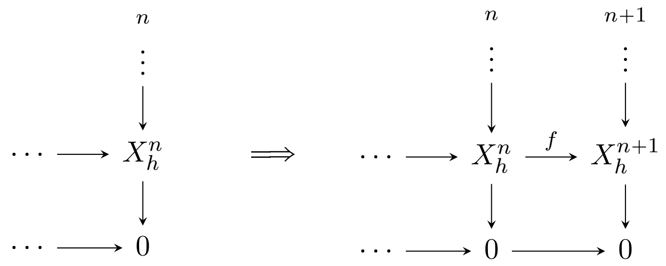
We illustrate now how to concatenate chain complexes and directed linear transformations. Recall the considerations about chain complexes concentrated in one or two degrees we made at the end of Chapter 1.

Construction 2.21. Under some circumstances we can concatenate a functor $X\colon [n] \to$ **ch** and a directed linear transformation (f, c) to form a new functor $X*(f, c) : [n+1] \to$ **ch**. Here is a description of this procedure and when it is allowed. Let the chain complex X^n be concentrated in degree h:

- Suppose that $c = r$ and the domain of f coincide with X^n_h. In this case, $X * (f, c) : [n+1] \to$ **ch** is given by the sequence of $n+1$ chain maps:

  $$\xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{S^h f}$$

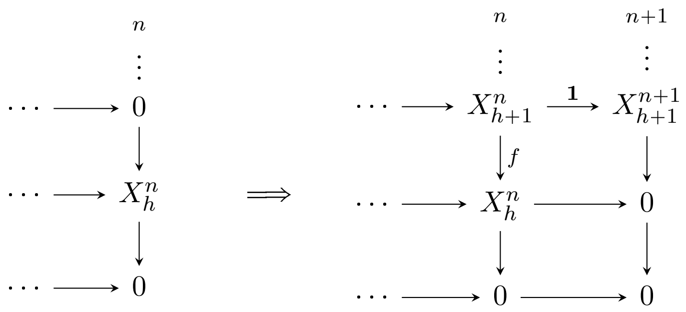
  where $S^h f$ is the h-th suspension of f. Graphically, this is depicted as

- Suppose that $c = l$ and the codomain of f coincides with X^n_h. In this case, put X^n_{h+1} equal to the domain of f, and $\partial^{X^n}_{h+1} = f$, so that X^n is now concentrated in degrees $h, h+1$. Note that we are modifying X^n. For $n > 0$, the transition morphism $X^{n-1<n}$ remains unchanged in degree h and it is set to zero otherwise. Then define Y as the chain complex concentrated in degree $h+1$, where it is equal to X^n_{h+1}, and the map $g\colon X^n \to Y$ as the chain map which is the identity in degree $h+1$ and zero otherwise. Define $X * (f, c) : [n+1] \to$ **ch** to be given by:

$\xrightarrow{g}$	if $n = 0$
$\xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{g}$	if $n > 0$

  Graphically, this is depicted as

■

Note that, in both cases, $(X * (f, c))^{n+1}$ is concentrated in one degree, and thus the construction is iterable.

The previous construction is a central passage to describe zigzag sequences as objects in $\mathbf{tame}\left([0,\infty),\mathbf{ch}\right)$. It required some choices, which we are going to motivate in Section 5.5.

Definition 2.22. The functor $X\colon [n] \to \mathbf{ch}$ is called a *discrete chain zigzag* if it satisfies the following requirements:

- X^i is either concentrated in one degree or two consecutive degrees for all $0 \leqslant i \leqslant n$;
- if X^i is concentrated in degree h, then X^{i+1} is either concentrated in degree h or it is concentrated in degrees $\{h, h+1\}$ for all $0 \leqslant i < n$;
- if X^i is concentrated in degrees $\{h, h+1\}$, then X^{i+1} is concentrated in degree $h+1$ and $X^{i<i+1} = \mathbf{1}$, for all $0 \leqslant i < n$.

Assume we are given a zigzag sequence $\{f_i\}$. We can use this information to inductively construct a discrete chain zigzag as follows.

Construction 2.23. Consider a zigzag sequence $\{f_i\}$. Let $X\colon [0] \to \mathbf{ch}$ be given by the chain complex concentrated in degree 0 for which:

$$X_0^0 = \begin{cases} \text{the domain of } f_1 & \text{if } c_1 = r \\ \text{the codomain of } f_1 & \text{if } c_1 = l \end{cases}$$

Assuming we have constructed $X\colon [i] \to \mathbf{ch}$, for $0 \leqslant i \leqslant m-1$, we can now define $X\colon [i+1] \to \mathbf{ch}$ by $X * (f_{i+1}, c_{i+1})$, according to Construction 2.21. We can then write the following discrete chain zigzag

$$X * (f_1, c_1) * \cdots * (f_m, c_m)$$

induced by the above data. ■

By Construction 2.23, from any zigzag sequence we obtain a discrete chain zigzag. The viceversa is also true: any discrete chain zigzag can be obtained from a zigzag sequence using Construction 2.23. Such a correspondence between zigzag sequences and discrete chain zigzags is bijective and preserves direct sums. A discrete chain zigzag given by an interval zigzag sequence is called *interval chain zigzag*.

We can now use the results of Section 2.1 to translate discrete chain zigzag into tame parametrised chain complexes. An object X in $\mathbf{tame}\left([0,\infty),\mathbf{ch}\right)$ is called a *zigzag* if it is isomorphic to the left Kan extension of a discrete chain zigzag $X\colon [n] \to \mathbf{ch}$ along some inclusion $[n] \subset [0,\infty)$. We use the symbol **ZigZag** to denote the full subcategory of $\mathbf{tame}\left([0,\infty),\mathbf{ch}\right)$ whose objects are zigzags. To illustrate how our construction works, consider the following zigzag sequence:

$$Z^0 \xrightarrow{f^1} Z^1 \xleftarrow{f^2} Z^2 \xrightarrow{f^3} Z^3 \xleftarrow{f^4} Z^4 \xleftarrow{f^5} Z^5$$

Such a zigzag sequence gives the following object in **ZigZag**:

$$\begin{array}{ccccccccccccc}
0 & & t_1 & & t_2 & & t_3 & & t_4 & & t_5 & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Z^5 & \xrightarrow{\mathbf{1}} & Z^5 & \xrightarrow{\mathbf{1}} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f^5} & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Z^4 & \xrightarrow{\mathbf{1}} & Z^4 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f^4} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^2 & \xrightarrow{\mathbf{1}} & Z^2 & \xrightarrow{f^3} & Z^3 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle f^2} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Z^0 & \xrightarrow{f^1} & Z^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where we used the shadowed squares to remark the nontrivial boundary maps. We call *interval zigzag*, denoted again by $\mathbb{Z}_{[t_s,t_e]}$, an object of **ZigZag** whose discrete chain zigzag is an interval chain zigzag $\mathbb{Z}_{[s,e]}$.

Once we fix a zigzag profile $\mathcal{C}$, we can use the bijective correspondence of Construction 2.23 to obtain the following corollary of Theorem 2.20, since the left Kan extension preserves direct sums.

Corollary 2.24. *Every object X in* **Zigzag** *decomposes uniquely up to isomorphisms into a finite direct sum of interval zigzags.*

Chapter 3

Parametrised objects in model categories

In this chapter, we introduce model category theory, and we show how to extend the model structure of a category $\mathcal{M}$ to the category $\mathbf{tame}([0,\infty), \mathcal{M})$ of tame parametrised objects of $\mathcal{M}$. We start recalling the definition and properties of a model category. Although the definitions and the results in Section 3.1 are classic (for references see [16, 36]), we decide to present all the proofs to keep the work self-contained. In Section 3.2, we recall an example of a model category, the category of compact chain complexes. Such an example is originally presented in [36]. In Section 3.3, we prove the main result of this chapter: the category of tame parametrised objects of a model category can be endowed with a model structure. This is a special case of the notion of a projective model structures on a tame subcategory of functor categories (for a reference, see for example [22], or chapter 11 of [23]). In the case of tame parametrised objects of a model category, such abstract structure can be explicitly described, and the axioms of a model category directly proven.

3.1 Introduction to model categories

Definition 3.1. A morphism $f\colon X_1 \to Y_1$ is called a *retract* of a morphism $g\colon X_2 \to Y_2$ if there is a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_1 \end{array}$$

where the composite top row $X_1 \to X_1$ is $\mathbf{1}$ and the composite bottom row $Y_1 \to Y_1$ is $\mathbf{1}$.

Definition 3.2. A model category structure on a category $\mathcal{M}$ is a choice of three distinguished classes of morphisms: weak equivalences (denoted by $\xrightarrow{\sim}$), fibrations (denoted by $\twoheadrightarrow$), and cofibrations (denoted by $\hookrightarrow$), such that the following axioms are satisfied:

A1. Finite limits and colimits exist in $\mathcal{M}$.

A2. If f and g are morphisms in $\mathcal{M}$ such that $g \circ f$ is defined, and if two of the three morphisms $f,\ g,\ g \circ f$ are weak equivalences, then so is the third (*two out of three property*).

A3. If f is a retract of g and g is a weak equivalence, fibration, or a cofibration, then so is f.

A4. In the two following solid commutative diagrams in $\mathcal{M}$ the dotted arrows exist and make the diagrams commute:

$$\begin{array}{ccc} X & \longrightarrow & Z \\ {\scriptstyle\sim}\big\downarrow & \nearrow & \big\downarrow \\ Y & \longrightarrow & W \end{array} \qquad \text{(S1)} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & Z \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle\sim} \\ Y & \longrightarrow & W \end{array} \qquad \text{(S2)}$$

A5. Every morphism $f : X \to Y$ in $\mathcal{M}$ can be factorised in two ways:

$$X \overset{c}{\underset{\sim}{\hookrightarrow}} X' \overset{p}{\twoheadrightarrow} Y, \quad p \circ c = f \qquad \text{(F1)} \qquad\qquad X \overset{c}{\hookrightarrow} Y' \overset{p}{\underset{\sim}{\twoheadrightarrow}} Y, \quad p \circ c = f \qquad \text{(F2)}$$

In the rest of the chapter, the symbol $\mathcal{M}$ denotes a category with a fixed model structure.

Axiom A1 guarantees the existence of initial and terminal objects, pushouts, pullbacks, and therefore of kernels and cokernels. Axiom A3, together with axiom A4, provides a characterisation of (co)fibrations. Axiom A4 states that morphisms that are cofibrations and weak equivalences have the left lifting property with respect to any fibration, and that morphisms that are fibrations and weak equivalences have the right lifting property with respect to any cofibration. Note that neither the lifts in axiom A4 nor the factorisations in axiom A5 are required to be unique. The axioms only require their existence.

We now characterise cofibrations and fibrations using axioms A3 and A4.

Lemma 3.3. *In $\mathcal{M}$, the following statements hold:*

1. *Cofibrations that are also weak equivalences are exactly the morphisms with left lifting property w.r.t. fibrations;*

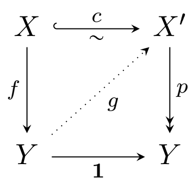
2. Cofibrations are exactly the morphisms with left lifting property w.r.t. fibrations that are also weak equivalences;

3. Fibrations that are also weak equivalences are exactly the morphisms with right lifting property w.r.t. cofibrations;

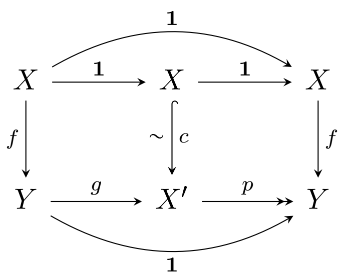
4. Fibrations are exactly the morphisms with right lifting property w.r.t. cofibrations that are also weak equivalences.

Proof. Axiom A4 implies that a (co)fibration that is also a weak equivalence has the right (left) lifting property with respect to any (co)fibration. We need to prove the converse. Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$.

(1) Suppose that f has the left lifting property w.r.t. to all fibrations. Factor f as in (F1). Then, by A4, there exists a morphism g, shown as dotted in the following diagram, such that the entire diagram commutes:

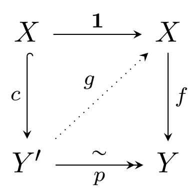


Hence, the following diagram commutes, i.e. f is a retract of c:

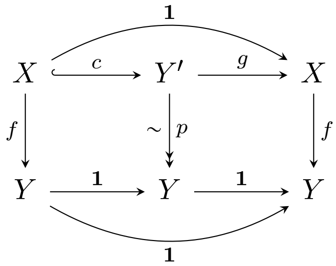


Then, by axiom A3, f is also a cofibration and a weak equivalence. To prove *(2)*, factor f as in (F2) and use the same argument.

(3) Suppose that f has the right lifting property w.r.t. to all cofibrations. Factor f as in (F2). Then, by lifting axiom A4, there exists a morphism g, shown dotted in the following diagram, such that the entire diagram commutes:



Hence, the following diagram commutes:

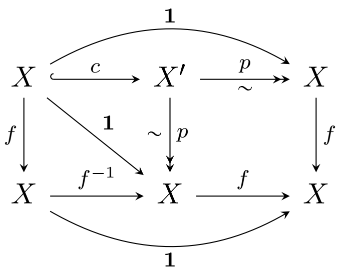


i.e. f is a retract of p. Then, by axiom A3, f is also a fibration and a weak equivalence. To prove *(4)*, factor f as in (F1) and use the same argument. □

There is a relation between weak equivalences and isomorphisms. All isomorphisms are weak equivalences, but the converse is not valid in general. Moreover, isomorphisms are also fibrations and cofibrations.

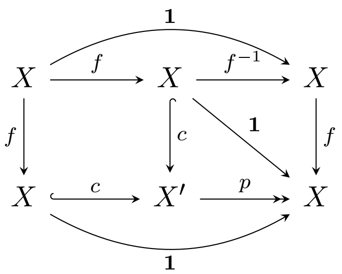
Proposition 3.4. *In $\mathcal{M}$, isomorphisms are weak equivalences, fibrations and cofibrations.*

Proof. Let $f\colon X \to X$ be an isomorphism in $\mathcal{M}$. We prove that f is a weak equivalence and a fibration by showing that it is the retract of a weak equivalence and a fibration. Consider the identity morphism $\mathbf{1}\colon X \to X$. By axiom A5, we have the factorisation $\mathbf{1} = p \circ c$, where c is a cofibration and p a fibration and a weak equivalence. Then the following diagram commutes:



Thus, by axiom A2, also f is a weak equivalence and a fibration.

We prove that f is a cofibration, showing that it is the retract of a cofibration. Consider the identity morphism $\mathbf{1}\colon X \to X$. By axiom A5, we have the factorisation $\mathbf{1} = p \circ c$, where c is a cofibration and p a fibration and a weak equivalence. Then the following diagram commutes:



Thus, by axiom A2, also f is a cofibration. □

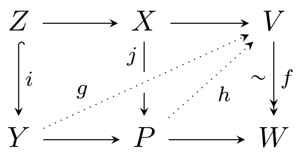
We know that, in a model category, pushouts and pullbacks exist. We are interested in how they interact with the distinguished classes of morphisms. As the following proposition shows, cofibrations are stable under pushout, and fibrations are stable under pullback. In general, weak equivalences are not stable under neither pushout or pullback. Here, we see a difference between isomorphisms and weak equivalences. Indeed, as shown in Proposition 1.16, isomorphisms are stable under pushout. If a morphism is not only a weak equivalence but also a (co)fibration, then it is stable under pullback (resp. pushout).

Proposition 3.5. *In $\mathcal{M}$, the following statements hold:*

1. *Cofibrations are stable under pushout;*
2. *Morphisms that are weak equivalences and cofibrations are stable under pushout;*
3. *Fibrations are stable under pullback;*
4. *Morphisms that are weak equivalences and fibrations are stable under pullback.*

Proof. We prove *(1)*. The proof of *(2)*, *(3)*, and *(4)* are similar.

By Lemma 3.3, it is enough to show that the pushout of a cofibration has the left lifting property w.r.t. morphisms that are weak equivalences and fibrations. Suppose we have the following commutative solid diagram:

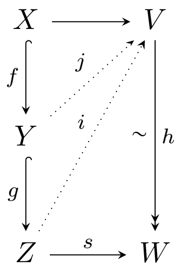


where f is a fibration and a weak equivalence, i is a cofibration and j is the pushout morphism. The morphisms $g\colon Y \to V$ is given by the left lifting property of i w.r.t. f. Since both X and Y map to V, by the universal property of pushout, there exists a morphism $h : P \to V$, such that the diagram commutes. Thus, the morphism j has the left lifting property w.r.t. morphisms that are fibrations and weak equivalences, proving the claim. □

From axiom A2, it follows in particular that the composition of two weak equivalences is a weak equivalence. The following proposition assures that also the composition of (co)fibrations is a (co)fibration. In general (co)fibrations do not have the two out of three property.

Proposition 3.6. *In $\mathcal{M}$, fibrations and cofibrations are stable under composition.*

Proof. Since the proof of the stability of fibrations is dual to the one of cofibrantions, we present only the details of the latter. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two cofibrations in $\mathcal{M}$ such that their composition $g \circ f$ exists. Let $h\colon V \to W$ be a fibration and a weak equivalence in $\mathcal{M}$ such that the solid diagram (3.1.1) commutes, for some suitable morphisms $X \to V$ and $s\colon Z \to W$. By Lemma 3.3, it is enought to show that $g \circ f$ has the left lifting property w.r.t. h.



(3.1.1)

In diagram (3.1.1), the morphism j is given by the left lifting property of f with respect to h. Then, by the left lifting property of g with respect to h, there exists a morphism $i\colon C \to V$ such that the whole diagram commutes. Hence, $g \circ f$ is a cofibration. □

Recall that a model category contains an initial and a terminal object. In general, they do not coincide. Initial and terminal objects are useful in model category theory because they are used to define fibrant and cofibrant objects. These are the objects whose behaviour with respect to weak equivalences is more controllable. Since they are more controllable, one can try to use them to study other objects.

Definition 3.7. An object X in $\mathcal{M}$ is called *fibrant* if the unique morphism $X \to *$ is a fibration.

Definition 3.8. An object X in $\mathcal{M}$ is called *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration.

3.2 The model category of compact chain complexes

In this section, we endow the category of compact chain complexes with the structure of a model category, and then we use such a structure to prove the standard decomposition of compact chain complexes.

In [16], Dwyer and Spaliński provide a model structure for **Ch**. Our interest is in the subcategory **ch** rather than in the whole **Ch**. Thus, we want to prove that, restricting the model structure defined by Dwyer and Spaliński on **Ch** to compact objects, we obtain a model structure on **ch**. This result is originally contained in [36], in a more abstract setting. Here, we present another, more explicit, proof. We now recall the definition of the three classes of morphisms from [16].

Definition 3.9. A morphism $f\colon X \to Y$ in **ch** is called:

- *weak equivalence* if $Hf_h\colon HX_h \to HY_h$ is an isomorphism for all degree $h \geqslant 0$;

- *fibration* if $f_h \colon X_h \to Y_h$ is an epimorphism for all degrees $h \geqslant 1$ (no assumption is made for $h = 0$);

- *cofibration* if $f_h \colon X_h \to Y_h$ is a monomorphism for all degrees $h \geqslant 0$.

To show that **ch** endowed with such a structure is a model category, we need to prove that the axioms A1-A5 are satisfied. Axioms A1-A4 are satisfied in **ch** as a direct consequence of the fact they are satisfied in **Ch** [16]. It remains to show that every morphism in **ch** factorises according to (F1) and (F2) through a compact chain complex. We prove it explicitly constructing two such factorisations.

Proposition 3.10. *Axiom A5 holds* **ch**.

Proof. Let X and Y be compact chain complexes, and $f : X \to Y$ a morphism between them. To get the factorisation in (F1), we use the path Construction 1.37. Take $X' \cong X \oplus PY$. Define the morphism $c \colon X \to X'$ as the inclusion into the first factor, and the morphism $p' \colon X' \to Y$ as $p' := \begin{bmatrix} f & p \end{bmatrix}$, where $p \colon PY \to Y$ is the morphism defined in Construction 1.37. The path of a compact chain complex is compact, and its direct sum with X is again compact. The morphism c is a monomorphism in all degrees, and thus a cofibration. Moreover, since $HPY = 0$, c induces an isomorphism in homology. Thus, c is also a weak equivalence. The morphism p' is an epimorphism in all degrees $h \geqslant 1$, and thus it is a fibration. By construction, $f = p' \circ c$, and (F1) holds.

To verify the factorisation (F2), we use the mapping cylinder Construction 1.38. Take $Y' \cong MC(f)$. Recall the morphisms i and p defined in Construction 1.38. p is a fibration and a weak equivalence. Define the morphism $c \colon X \to MC(f)$ as $c := \begin{bmatrix} i \\ f \end{bmatrix}$. By construction, c is a cofibration and $f = p \circ c$. The mapping cylinder of a morphisms between compact chain complexes is a compact chain complex. This proves the claim. □

Corollary 3.11. **ch***, with the model structure defined in Definition 3.9, is a model category.*

From now on, we use the symbol **ch** to denote the category of compact chain complexes endowed with the model structure of Definition 3.9.

Remark 3.12. In **ch**, there is the zero object, the chain complex zero in each degree. Then, from the definition of fibrations and cofibrations in **ch**, it follows that all compact chain complexes are both cofibrant and fibrant. Indeed, all objects receive a monomorphism from the trivial chain complex, and all objects map epimorphically onto it.

Standard decomposition of ch

In this subsection, we use the model category setting to prove the standard decomposition of a chain complex. This result is already know in the literature (see for example [25]), but here it is presented in the setting of model category. Let X be a chain complex. Use Construction 1.37 to build two chain complexes out of it, denoted by DBX and SBX. Recall that the chain complex BX is the graded vector space of the boundaries of X. SBX is the suspension of BX, and DBX is the cone over BX, given by:

$$\cdots \xrightarrow{\begin{bmatrix}0 & \mathbf{1}\\ 0 & 0\end{bmatrix}} DB_2X \oplus DB_1X \xrightarrow{\begin{bmatrix}0 & -\mathbf{1}\\ 0 & 0\end{bmatrix}} DB_1X \oplus DB_0X \xrightarrow{\begin{bmatrix}0 & \mathbf{1}\end{bmatrix}} DB_0X$$

Recall the epimorphism $p\colon DBX \to SBX$ defined in Construction 1.35. Since in **ch** all epimorphisms are fibrations, p is a fibration. Consider then $\partial\colon X \to SBX$, the boundary morphism of X. Also $\partial\colon X \to SBX$ is an epimorphism in all degrees, and, thus, also ∂ is a fibration. The morphism $0 \overset{\sim}{\hookrightarrow} DBX$ is a cofibration since all the objects in **ch** are cofibrant, and a weak equivalence, since $H_hDBX = 0$ for all $h = \mathbb{N}$. Hence, axiom A4 guarantees the existence of a morphism $\varphi\colon DBX \to X$ making the following diagram commute:

$$\begin{array}{ccc}
0 & \longrightarrow & X \\
{\scriptstyle\sim}\big\downarrow & \overset{\varphi}{\nearrow} & \big\downarrow{\scriptstyle\partial} \\
DBX & \xrightarrow{\;p\;} & SBX
\end{array}$$

The restriction of any such φ to $i\colon BX \hookrightarrow DBX$ is the standard inclusion $BX \hookrightarrow ZX \hookrightarrow X$, where i is the morphism defined in Construction 1.35. To see it, consider the morphism $\psi\colon BX \to ZX$ such that the following diagram commutes, whose existence is ensured by Proposition 1.31:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & BX & \xhookrightarrow{\;i\;} & DBX & \xrightarrow{\;p\;} & SBX & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\mathbf{1}} & & \\
0 & \longrightarrow & ZX & \hookrightarrow & X & \xrightarrow{\;\partial\;} & SBX & \longrightarrow & 0
\end{array} \tag{3.2.1}$$

Since $H_hBX = B_hX$, $H_hZX = Z_hX$, $H_hSBX = B_{h-1}X$, and $H_hDBX = 0$, applying the long exact sequence in homology (Theorem 1.39) to the rows of (3.2.1) gives:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & B_hX & \xrightarrow{\;\delta\;} & B_hX & \longrightarrow & 0 & \longrightarrow & B_{h-1}X & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle\mathbf{1}} & & \big\downarrow{\scriptstyle\psi} & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & H_{h+1}X & \xrightarrow{\;0\;} & B_hX & \xhookrightarrow{\;\delta'\;} & Z_hX & \twoheadrightarrow & H_hX & \xrightarrow{\;0\;} & B_{h-1}X & \longrightarrow & \cdots
\end{array}$$

where δ and δ' are the connecting homomorphisms. The induced homomorphism in homology $H\psi$ is ψ itself, since $HBX = BX$ and $HZX = ZX$. Moreover, we have

$\delta = \mathbf{1}$. A direct computation of the connecting homomorphisms (1.3.1) shows that δ' is the standard inclusion. Then, by commutativity of the the diagram, also ψ is the standard inclusion. Finally, by commutativity of (3.2.1) the restriction of φ to i is the standard inclusion. By Proposition 1.33, the morphism φ leads therefore to a pushout square:

$$\begin{array}{ccc} BX & \xhookrightarrow{i} & DBX \\ \big\downarrow & & \big\downarrow \varphi \\ ZX & \hookrightarrow & X \end{array}$$

Since $BX \hookrightarrow ZX$ is a monomorphism and thus a cofibration in **ch**, by Proposition 3.5, the morphism φ is a cofibration, as depicted in the diagram. Since $\mathtt{k}$ is a field, and all the differentials in BX, ZX and HX are trivial, there exists a chain map $s\colon HX \to ZX$ whose composition with the quotient $ZX \twoheadrightarrow HX$ is the identity on HX. For any such s, the morphism $\begin{bmatrix} i & s \end{bmatrix} : BX \oplus HX \to ZX$ is an isomorphism. To see it, note that the morphism s is a section, and thus the short exact sequence $BX \hookrightarrow ZX \twoheadrightarrow HX$ splits by Proposition 1.28. Hence, the morphism $\begin{bmatrix} i & s \end{bmatrix} : BX \oplus HX \to ZX$ is an isomorphism. We claim that also the morphism $\begin{bmatrix} \varphi & s \end{bmatrix} : DBX \oplus HX \to X$ is an isomorphism, where the symbol s denotes the composition of $s\colon HX \to ZX$ and the inclusion $ZX \hookrightarrow X$. The claim follows from the fact that isomorphisms are stable under pushout (Proposition 1.16), and that X is the pushout of $\begin{bmatrix} i & \mathbf{1} \end{bmatrix} : BX \oplus HX \to DBX \oplus HX$ and $\begin{bmatrix} i & s \end{bmatrix} : BX \oplus HX \to ZX$.
We call $DBX \oplus HX$ the *standard decomposition* of the chain complex X.

Every chain complex decomposes into direct sum of disks and spheres. Indeed, HX has all trivial differentials, and thus $HX \cong \oplus_h \bigoplus_{i=0}^{\dim H_h X} S^h$. On the other hand, by definition of DBX (Construction 1.35), we have $DBX \cong \oplus_h \bigoplus_{i=0}^{\dim B_h X} D^h$. Since S^h and D^h are indecomposable, by Remark 2.17 such a decomposition is unique up to isomorphisms. Moreover, as a direct corollary of the decomposition, we get the following property.

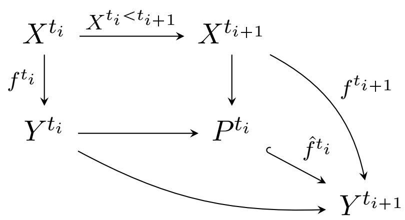
Corollary 3.13. *A chain complex has trivial homology if and only if it is a direct sum of disks.*

3.3 Model category of tame parametrised objects

In this section, we prove one of our main result: the category of tame parametrised objects of a model category admits a model category structure.

Definition 3.14. Let X and Y be objects in $\mathbf{tame}([0,\infty),\mathcal{M})$, and $0 = t_0 < t_1 < \cdots < t_n$ in $[0,\infty)$ a sequence discretising both of them. A morphism $f\colon X \to Y$ in $\mathbf{tame}([0,\infty),\mathcal{M})$ is

- a weak equivalence if $f^t\colon X^t \to Y^t$ is a weak equivalence in $\mathcal{M}$, for every $t \in [0,\infty)$;
- a fibration if $f^t\colon X^t \to Y^t$ is a fibration in $\mathcal{M}$, for every $t \in [0,\infty)$;
- a cofibration if
  (i) $f^0\colon X^0 \to Y^0$ is a cofibration in $\mathcal{M}$;
  (ii) $\hat{f}^{t_i}\colon P^{t_i} \to Y^{t_{i+1}}$ is a cofibration in $\mathcal{M}$ for all $i = 0,\ldots,n-1$, where $\hat{f}^{t_i}$ is the mediating morphism of the pushout of $X^{t_i<t_{i+1}}$ and f^{t_i}, depicted in the following diagram:



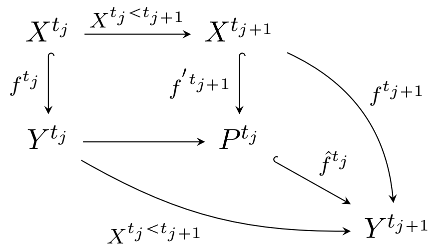
In what follows, we use ˆ to indicate the mediating morphism of a pushout. Note that ˆ is functorial.

We verify axioms A1-A3. Recall that limits and colimits are formed objectwise in functor categories, and that any finite collection of objects in $\mathbf{tame}([0,\infty),\mathcal{M})$ admits a common discretising sequence. Then axiom A1 is verified in $\mathbf{tame}([0,\infty),\mathcal{M})$ since all finite limits and colimits exist in $\mathcal{M}$. Axiom A2 is valid in $\mathbf{tame}([0,\infty),\mathcal{M})$ because it is verified objectwise and $\mathcal{M}$ is a model category. Axiom A3 is verified for weak equivalences and fibrations because it is verified pointwise and $\mathcal{M}$ is a model category. Axiom A3 is verified for cofibrations by functoriality of ˆ, and by the fact that $\mathcal{M}$ is a model category.

The converse to the following lemma is not valid in general.

Lemma 3.15. *If $f\colon X \to Y$ is a cofibration in* $\mathbf{tame}([0,\infty),\mathcal{M})$*, then $f^t\colon X^t \to Y^t$ is a cofibration in $\mathcal{M}$ for every t in $[0,\infty)$.*

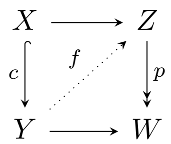
Proof. Let $0 = t_0 < \cdots < t_n$ be a sequence that discretises both X and Y. We prove the claim by induction on the steps of the sequence. For t_0, the claim holds by definition of cofibration in $\mathbf{tame}([0,\infty),\mathcal{M})$. Suppose now the claim holds for all t_i, for all $i = 0,\ldots,j$. We show it is also true in $j+1$. Consider the following pushout diagram:



where f^{t_j} is a cofibration by inductive hypothesis, ${f'}^{t_{j+1}}$ is a cofibration since it is the pushout of a cofibration in $\mathcal{M}$ (Proposition 3.5), and $\hat{f}^{t_j}$ is a cofibration by definition of cofibrations in $\mathcal{M}$. Since the composition of cofibrations is a cofibration (Proposition 3.6), $f^{t_{j+1}}$ is a cofibration. □

We are now ready to show that axiom A4 is satisfied. We use the previous lemma as key argument.

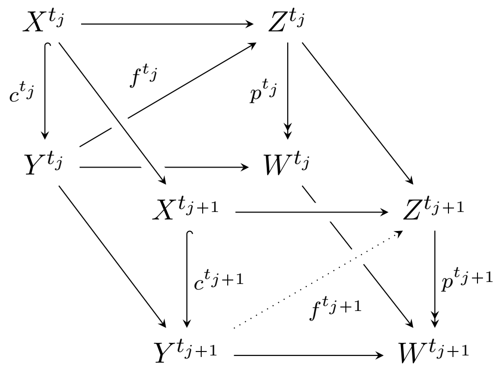
Proposition 3.16. *Let $c\colon X \to Y$ and $p\colon Z \to W$ be morphisms in* **tame** $([0,\infty), \mathcal{M})$, *such that the following solid diagram commutes:*



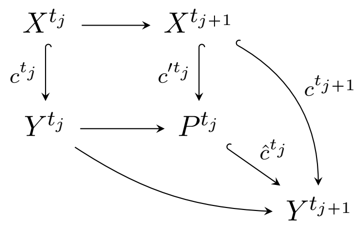
where c is a cofibration, p is a fibration, and one of them is also a weak equivalence. Then the lift morphism $f\colon Y \to Z$ exists, making the diagram commute.

Note that, by the model structure on $\mathcal{M}$ and Lemma 3.15, such a lift exists when we restrict on any t in $[0,\infty)$. We need to find a compatible lift for all t in $[0,\infty)$.

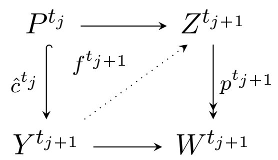
Proof. Let $0 = t_0 < t_1 < \cdots < t_n$ be a common discretising sequence of X, Y, Z and W. We construct a compatible lift f by induction on the sequence. At step 0, let $f^0\colon Y^0 \to Z^0$ be a lift given by axiom A4 in $\mathcal{M}$. Suppose we have constructed a compatible family of lifts $\{f^{t_i}\}$ for $i = 0, \dots, n$. We now construct a compatible lift $f^{t_{j+1}}$. Consider the following solid diagram:



where c^{t_j} and $c^{t_{j+1}}$ are cofibrations by Lemma 3.15. Take the following pushout:



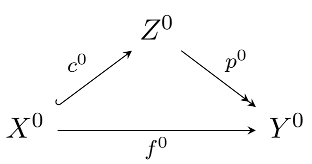
where c'^{t_j} is a cofibration being a pushout of a cofibration (Proposition 3.5). By the same proposition, c'^{t_j} is also a weak equivalence when c^{t_j} is a weak equivalence. The morphism $\hat{c}^{t_j}$ is a cofibration by Definition 3.14. Moreover, whenever c^{t_j} and $c^{t_{j+1}}$ are weak equivalences, by axiom A2 also $\hat{c}^{t_j}$ is a weak equivalence. Since both $X^{t_{j+1}}$ and Y^{t_j} map to $Z^{t_{j+1}}$, by the universal property of pushout there is a map $P^{t_j} \to Z^{t_{j+1}}$. Applying the A4 to the solid square



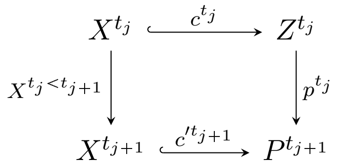
we obtain the compatible lift $f^{t_{j+1}}$. □

Consider the morphism $f\colon X \to Y$ in the model category **tame** $([0,\infty), \mathcal{M})$. We want to build a general construction to iteratively find a factorisation $f = p \circ c$, where c is a cofibration, p is a fibration and one of them is also a weak equivalence. This construction is providing one of the possible factorisations. The reason why we opted to present this factorisation is because it is useful in the construction of the invariants in **tame** $([0,\infty), \mathbf{ch})$ in Chapter 5. Note that, by the model structure on $\mathcal{M}$ and Lemma 3.15, the factorisations exists when we restrict to any t in $[0,\infty)$. Our goal is to explain why it is possible to choose such factorisations to be compatible.

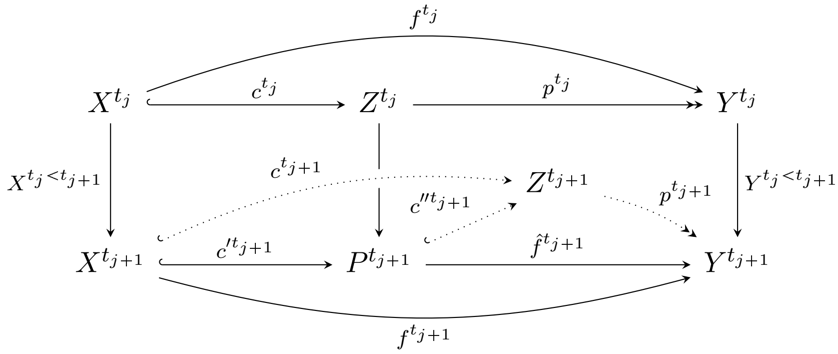
Construction 3.17 (Iterative factorisation)**.** Let $0 = t_0 < \cdots < t_n$ be a sequence discretising both X and Y. We build the factorisations inductively. By axiom A5 in $\mathcal{M}$, for $t_0 = 0$ there exists a factorisation of $f^0\colon X^0 \to Y^0$:



where c^0 is a cofibration, p^0 is a fibration and one of them is also a weak equivalence. Assume that we have defined c^{t_i}, p^{t_i} and Z^{t_i} for all $i \leqslant j$, such that c^{t_i} is cofibration, p^{t_i} fibration and one of them, consistenly for each i, is also a weak equivalence. Then we can use the following construction to build $c^{t_{j+1}}$, $p^{t_{j+1}}$ and $Z^{t_{j+1}}$. First, define P^{t_j} to be the pushout of c^{t_j} and $X^{t_j < t_{j+1}}$ in the following diagram:



Next, consider the following commutative solid diagram:



where the left square is the previously defined pushout and the morphism $\hat{f}^{t_{j+1}}$ is the unique mediating morphism induced by the universal property of the pushout. The morphisms $c''^{t_{j+1}}$ and $p^{t_{j+1}}$ form a factorisation of $\hat{f}^{t_{j+1}}$ and are given by axiom A5 in $\mathcal{M}$. $Z^{t_{j+1}}$ and $p^{t_{j+1}}$ are respectively the object and one of the morphism we aimed to build. The other morphism is $c^{t_{j+1}}$, given by the composition $c''^{t_{j+1}} \circ c'^{t_{j+1}}$. Note that, by Proposition 3.5, $c'^{t_{j+1}}$ is a cofibration. Thus, by Proposition 3.6, $c^{t_{j+1}}$ is a cofibration. Moreover, if c^{t_j} is a weak equivalence, by Proposition 3.5 also $c'^{t_{j+1}}$ is weak equivalence. Choosing $c''^{t_{j+1}}$ to be a weak equivalence, by Proposition 3.6 also $c^{t_{j+1}}$ becomes a weak equivalence. On the other hand, if p^{t_j} is a weak equivalence, we can choose $p^{t_{j+1}}$ to be a weak equivalence. Then c^{t_i} and p^{t_i} are compatible factorisation of f^{t_i}, for $i = 0, \ldots j + 1$.

Finally, define Z to be the left Kan extension along $0 = t_0 < \cdots < t_n$ of the sequence $\left\{Z^{t_i<t_{i+1}}\right\}_{0\leqslant i\leqslant n-1}$. Let $c\colon X \to Z$ and $p\colon Z \to Y$ be the natural transformations induced by the sequences of morphisms $\left\{c^{t_i}\colon X^{t_i} \to Z^{t_i}\right\}_{1\leqslant i\leqslant n}$ and $\left\{p^{t_i}\colon Z^{t_i} \to Y^{t_i}\right\}_{1\leqslant i\leqslant n}$. Then Z, c and p give the wanted factorisations. ■

We can now prove axiom A5 in $\mathbf{tame}\left([0, \infty), \mathbf{ch}\right)$. The above construction shows that any morphism in $\mathbf{tame}\left([0, \infty), \mathcal{M}\right)$ factorises as in (F1) and (F2).

Corollary 3.18 (A5)**.** *The morphisms of Definition 3.14 in* $\mathbf{tame}\left([0, \infty), \mathcal{M}\right)$ *satisfy axiom A5.*

We then obtain as a corollary the main results of this chapter: $\mathbf{tame}\left([0, \infty), \mathcal{M}\right)$ admits a model category structure. In Chapter 4 and Chapter 5, we make extensive use of such a model category structure in the particular case of $\mathbf{tame}\left([0, \infty), \mathbf{ch}\right)$, to retrieve invariants describing tame parametrised chain complexes.

Corollary 3.19. $\mathbf{tame}([0,\infty), \mathcal{M})$, *endowed with the structure described in Definition 3.14, is a model category.*

For the rest of the work, we use the symbol $\mathbf{tame}([0,\infty), \mathcal{M})$ to denote the category of tame parametrised objects of $\mathcal{M}$ endowed with the model structure of Definition 3.14.

Chapter 4

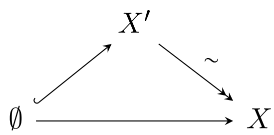
Decomposition of cofibrant objects

The goal of this chapter is to prove a decomposition theorem for cofibrant objects in $\mathbf{tame}([0,\infty), \mathbf{ch})$. We have more control over cofibrant objects than over other objects. This is well displayed in this chapter because, despite $\mathbf{tame}([0,\infty), \mathbf{ch})$ is of wild representation type (Section 2.5 - Commutative ladders), we can fully describe the indecomposables of cofibrant objects.

The chapter is structured as follows. In Section 4.1, we describe the cofibrant replacement in model category. Next, we characterise the cofibrant objects in general in $\mathbf{tame}([0,\infty), \mathcal{M})$. In Section 4.2, we characterise the cofibrations in $\mathbf{tame}([0,\infty), \mathbf{ch})$. Finally, in Section 4.3, we present the decomposition theorem for cofibrant objects in $\mathbf{tame}([0,\infty), \mathbf{ch})$.

4.1 Cofibrant objects in $\mathbf{tame}([0,\infty), \mathcal{M})$

We begin introducing the notion of cofibrant replacement. For every object X in a model category $\mathcal{M}$, there exists at least one cofibrant object X' that maps to X as a weak equivalence and a fibration $X' \overset{\sim}{\twoheadrightarrow} X$. Indeed, consider an object X in $\mathbf{tame}([0,\infty), \mathcal{M})$ and factorise the unique morphism $\emptyset \to X$ with (F2). Any fibration and weak equivalence that fits into the following diagram is a *cofibrant replacement*.



For documentation about cofibrant replacement, see for example [21]. Note that the isomorphism type is not determined, and there are many not isomorphic cofibrant replacements of an object.

We next characterise the cofibrant objects in $\mathbf{tame}([0,\infty), \mathcal{M})$.

Proposition 4.1. *Let $0 = t_0 < \cdots < t_n$ be a sequence that discretises an object X in $\mathbf{tame}([0,\infty), \mathcal{M})$. Then X is cofibrant if and only if X^0 is cofibrant in $\mathcal{M}$ and the transition morphism $X^{t_i < t_{i+1}} \colon X^{t_i} \to X^{i+1}$ is a cofibration for every $i = 0, \ldots, n-1$.*

Proof. From Definition 3.14 of cofibrations in $\mathbf{tame}([0,\infty), \mathcal{M})$, it follows that if the morphism $f \colon \emptyset \hookrightarrow X$ is a cofibration, then X^0 is cofibrant, since it receives a cofibration from $\emptyset$. Thus, X^0 being cofibrant is a necessary condition for $f \colon \emptyset \hookrightarrow X$ to be a cofibration.

Let $0 = t_0 < \cdots < t_n$ be a sequence that discretises X. By definition, $\emptyset \to X$ is a cofibration if $\emptyset \to X^0$ and $P^{t_i} \to X^{t_{i+1}}$ are cofibrations in $\mathcal{M}$ for all $i = 0, \ldots, n-1$. In this case $P^0 = \emptyset, P^{t_1} = X^0, \ldots, P^{t_n} = X^{t_{n-1}}$, and the morphisms $P^{t_i} = X^{t_{i-1}} \to X^{t_i}$ are the transition morphisms in X. The claim follows. □

Note that, in the above proposition, we made no assumptions on $\mathcal{M}$. Thus, the characterisation we presented holds in the general case of $\mathbf{tame}([0,\infty), \mathcal{M})$. Since we are interested in $\mathbf{tame}([0,\infty), \mathbf{ch})$, we would like to study further the cofibrations and the cofibrant objects therein. To start, note that, since in **ch** the cofibrations are given by the monomorphisms, Proposition 4.1 is stating that the cofibrant objects in $\mathbf{tame}([0,\infty), \mathbf{ch})$ are the tame parametrised chain complexes whose transition morphisms are monomorphisms.

4.2 Characterisation of cofibrations in tame([0, ∞), ch)

By Lemma 3.15, we know that a cofibration in $\mathbf{tame}([0,\infty), \mathbf{ch})$ is pointwise a cofibration in **ch**. This is not enough for a morphism to be a cofibration. Here, we prove some useful characterisations of cofibrantions. These characterisations are fundamental in the proof of the decomposition theorem of cofibrant objects. Recall that $\mathbf{tame}([0,\infty), \mathbf{ch})$ and **ch** are abelian categories as well as model categories.

Proposition 4.2. *The following statements are equivalent for a morphism $f \colon X \to Y$ in $\mathbf{tame}([0,\infty), \mathbf{ch})$:*

(i) f is a cofibration;

*(ii) For all t in $[0,\infty)$, $f^t \colon X^t \to Y^t$ is a cofibration in **ch**, and $Y/f(X)$ is cofibrant;*

*(iii) For all t in $[0,\infty)$, $f^t \colon X^t \to Y^t$ is a cofibration in **ch**, and, for all $s < t$ in $[0,\infty)$, the following is a pullback square:*

$$\begin{array}{ccc} X^s & \overset{f^s}{\hookrightarrow} & Y^s \\ {\scriptstyle X^{s<t}}\downarrow & & \downarrow{\scriptstyle Y^{s<t}} \\ X^t & \underset{}{\overset{f^t}{\hookrightarrow}} & Y^t \end{array}$$

Proof. *(i)* $\Longrightarrow$ *(ii)*
The morphisms f^t are monomorphisms for all t in $[0, \infty)$ by Lemma 3.15. By Definition 1.19, we have the following pushout square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y/f(X) \end{array}$$

If f is a cofibration, then also $0 \to Y/f(X)$ is a cofibration by Proposition 3.5, and thus $Y/f(X)$ is cofibrant.
(ii) $\Longrightarrow$ *(iii)*

For all $s < t$ in $[0, \infty)$, we have the following commutative diagram with exact rows:

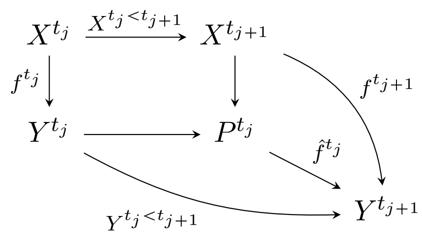
$$\begin{array}{ccccc} X^s & \overset{f^s}{\hookrightarrow} & Y^s & \twoheadrightarrow & Y^s/f(X)^s \\ {\scriptstyle X^{s<t}}\downarrow & & \downarrow{\scriptstyle Y^{s<t}} & & \downarrow \\ X^t & \overset{f^t}{\hookrightarrow} & Y^t & \twoheadrightarrow & Y^t/f(X)^t \end{array} \tag{4.2.1}$$

where the morphism $Y^s/f(X)^s \to Y^t/f(X)^t$ is a cofibration by Proposition 4.1. In particular, such a morphism is a monomorphism. Our goal is to prove that the left inner square in (4.2.1) is a pullback. Consider then the following commutative diagram where both inner squares are pullback:

$$\begin{array}{ccccc} P & \longrightarrow & X^t & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f^t} & & \downarrow \\ Y^s & \xrightarrow{Y^{s<t}} & Y^t & \longrightarrow & Y^t/f(X)^t \end{array}$$

Then, by the pasting law of pullback (Proposition 1.17), also the outer diagram is a pullback. It follows that P is isomorphic to the kernel of the composition of $Y^{s<t}$ and $Y^t \twoheadrightarrow Y^t/f(X)^t$. Since $Y^s/f(X)^s \longrightarrow Y^t/f(X)^t$ is a monomorphism, the kernel of the composition $Y^s \xrightarrow{Y^{s<t}} Y^t \twoheadrightarrow Y^t/f(X)^t$ coincides with the kernel of $Y^s \twoheadrightarrow Y^s/f(X)^s$ which, by exactness of the row, is X^s. Consequently, the left square of (4.2.1) is a pullback and the claim is proved.
(iii) $\Longrightarrow$ *(i)*

Let $0 = t_0 < t_1 < \cdots < t_n$ be a sequence discretising both X and Y. We need to show that $\hat{f}^{t_i} \colon P^{t_i} \to Y^{t_i}$ is a cofibration for all $i = 0, \ldots, n$. We prove the claim by induction on i. The morphism $f^0 \colon X^0 \to Y^0$ is a cofibration by assumption. Assume now that the claim is true for all $i = 0, \ldots, j$. We show it is valid also at step $i = j + 1$. Consider the following pushout diagram:



Since by hypothesis the outer square is a pullback, its mediating morphism is the identity. Then, applying the Lemma 1.34, we obtain that the morphism $\hat{f}^{t_j}: P^{t_j} \to Y^{t_{j+1}}$ is a monomorphism, i.e. a cofibration in **ch**. This proves the claim. $\square$

4.3 Decomposition of cofibrant objects in tame $([0,\infty), \mathbf{ch})$

In this section, we prove the decomposition theorem of cofibrant objects. As shown in Proposition 4.1, the transition morphisms of a cofibrant object are monomorphisms. In the literature, such objects are known as filtered chain complexes. Their structure theorem has been already proven in different contexts [5, 30, 44, 45], but never, at the best of our knowledge, in the model category setting.

To prove the decomposition theorem, we split the interval sphere according to a specific order. Such an order is given by Proposition 4.2, studying the cofibration out of interval spheres using the pullback. To explain how to enumerate such cofibrations, we analyse first how morphisms out of an interval sphere behave. We begin by studying morphisms out of $I^h[b,\infty)$.

Remark 4.3. Let b be in $[0,\infty)$. A morphism $f: I^h[b,\infty) \to X$ leads to a linear transformation $f_h^b: I^h[b,\infty)_h^b = \mathtt{k} \to X_h^b$. Let $x := f_h^b(1)$ in X_h^b. Note that x satisfies the equation $\partial(x) = 0$ since $I^h[b,\infty)$ is concentrated in degree h. This means that x belongs to the cycles ZX_h^b. Choosing an element in ZX_h^b is all what is needed to describe a morphism out of $I^h[b,\infty)$. For any x in ZX_h^b, there is a unique morphism $I(x): I^h[b,\infty) \to X$ such that $x = I(x)_h^b(1)$. The association $f \mapsto f_h^b(1)$ describes a bijection between the set of morphisms $\mathbf{Hom}_{\mathbf{tame}([0,\infty),\mathbf{ch})}\left(I^h[b,\infty), X\right)$ and the set of cycles $Z_h X^b$.

We study now the morphisms out of $I^h[b,d)$.

Remark 4.4. Let $b \leqslant d$ be in $[0,\infty)$. A morphism $f: I^h[b,d) \to X$ leads to two linear morphisms

$$f_h^b: I^h[b,d)_h^b = \mathtt{k} \to X_h^b$$
$$f_{h+1}^b: I^n[b,d)_{h+1}^d = \mathtt{k} \to X_{h+1}^d$$

Define two elements $x := f_h^b(1)$ in X_h^b and $y := f_{h+1}^d(1)$ in X_{h+1}^d. The elements x and y satisfy the following equations.

$$\partial(x) = 0$$
$$X_h^{b\leqslant d}(x) = \partial(y)$$

These equations contain all the information needed to describe a morphism from $I^h[b,d)$. Moreover, if x in X_h^b and y in X_{h+1}^d satisfy the above equations, then there is a unique natural transformation $I(x,y)\colon I^h[b,d) \to X$ such that $x = I(x,y)_h^b(1)$ and $y = I(x,y)_{h+1}^d(1)$. Therefore, the association $f \mapsto \left(f_h^b(1), f_{h+1}^d(1)\right)$ describes an isomorphism between the set of morphisms $\mathbf{Hom}_{\mathbf{tame}([0,\infty),\mathbf{ch})}\left(I^h[b,d), X\right)$ and the pullback P:

$$\begin{array}{ccc} P & \longrightarrow & X_{h+1}^d \\ \downarrow & & \downarrow \partial \\ ZX_h^d & \xrightarrow{X_h^{b\leqslant d}} & X_h^d \end{array}$$

Our next step it to characterise cofibrations out of $I^h[b,d)$ and $I^h[b,\infty)$.

Proposition 4.5. *Let X be an object in* $\mathbf{tame}([0,\infty),\mathbf{ch})$.

1. *Consider $h \in \mathbb{N}$ and b in $[0,\infty)$. Choose x in Z_hX^b. Then $I(x) : I^h[b,\infty) \to X$ is a cofibration if and only if X is cofibrant and x is not in the image of $X_h^{t<b}$ for any $t < b$;*

2. *Consider $h \in \mathbb{N}$ and b in $[0,\infty)$. Choose x in Z_hX^b and y in X_{h+1}^d such that $X_h^{b\leqslant d}(x) = \partial(y)$. Then $I(x,y) : I^h[b,d) \to X$ is a cofibration if and only if X is cofibrant, x is not in the image of $X_h^{s<b}$ for any $s < b$, and y is not in the image of $X_{h+1}^{t<d}$ for any $t < d$.*

Proof. We prove the two claims together. With a little abuse of notation, we are going to use the symbol $I^h[b,d)$ also for the case $d = \infty$.

First, note that, since $I^h[b,d)$ is cofibrant, if $f\colon I^h[b,d) \to X$ is a cofibration then X has to be necessary cofibrant. This because any object receiving a cofibration from a cofibrant object is cofibrant. For the rest of the proof, then, we assume X cofibrant. This, by Proposition 4.1, implies that the transition morphisms of X are cofibrations.

By extending any sequence that discretises X with elements b and d, if $d < \infty$, we get a sequence that discretises both X and $I^h[b,d)$. Let $0 = t_0 < t_1 < \cdots < t_n$ be such an extended sequence. Consider the diagram:

$$\begin{array}{ccc} I^h\left[b,d\right)^{t_j} & \xrightarrow{I^h[b,d)^{t_j<t_{j+1}}} & I^h\left[b,d\right)^{t_{j+1}} \\ {\scriptstyle f^{t_j}}\downarrow & & \downarrow{\scriptstyle f^{t_{j+1}}} \\ X^{t_j} & \xrightarrow{X^{t_j<t_{j+1}}} & X^{t_{j+1}} \end{array} \tag{4.3.1}$$

By Proposition 4.2, since X is cofibrant, the morphism $f\colon I^h\left[b,d\right) \to X$ is a cofibration if and only if the diagram (4.3.1) is a pullback for all $j = 0,\ldots,n-1$. Since the transition morphisms of $I^h\left[b,d\right)$ are 0- or 1-dimensional, we can enumerate the cases in which they change and study the diagram (4.3.1) in each of them. If $d < \infty$, and $t_j > d$, in degrees $l = h, h+1$, and if $b \leqslant t_j < t_{j+1} < d$ in degree $l = h$, (4.3.1) becomes

$$\begin{array}{ccc} \mathtt{k} & \longrightarrow & \mathtt{k} \\ \downarrow & & \downarrow \\ X_l^{t_j} & \xrightarrow{X_l^{t_j<t_{j+1}}} & X_l^{t_{j+1}} \end{array}$$

and hence it is a pullback square. If $t_j < b = t_{j+1}$, in degree h (4.3.1) becomes

$$\begin{array}{ccc} 0 & \longrightarrow & \mathtt{k} \\ \downarrow & & \downarrow{\scriptstyle f^{t_{j+1}}} \\ X_h^{t_{j+1}} & \xrightarrow{X^{t_j<t_{j+1}}} & X_h^{t_{j+1}} \end{array}$$

Thus, it is a pullback if and only if $f^{t_{j+1}}(x)$ is not in the image of $X^{t_j<t_{j+1}}$. If $d < \infty$ and $t_j < d = t_{j+1}$, in degrees h and $h+1$ (4.3.1) becomes

$$\begin{array}{ccc} \mathtt{k} & \longrightarrow & \mathtt{k} \\ \downarrow & & \downarrow \\ X_h^{t_{j+1}} & \xrightarrow{X_h^{t_j<t_{j+1}}} & X_h^{t_{j+1}} \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & \mathtt{k} \\ \downarrow & & \downarrow{\scriptstyle f_{h+1}^{t_{j+1}}} \\ X_{h+1}^{t_{j+1}} & \xrightarrow{X_{h+1}^{t_j<t_{j+1}}} & X_{h+1}^{t_{j+1}} \end{array}$$

The left diagram is pullback square, and the right diagram is a pullback square if and only if $f_{h+1}^{t_{j+1}}(y)$ is not in the image of $X_{h+1}^{t_j<t_{j+1}}$. In all other degrees and combinations of b and d with the discretising sequence, (4.3.1) becomes

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_l^{t_j} & \xrightarrow{X_l^{t_j<t_{j+1}}} & X_l^{t_{j+1}} \end{array}$$

for $l \in \mathbb{N}$. Hence, it is a pullback square, and the claim is proved. □

We are now ready to prove the decomposition theorem of cofibrant objects.

Theorem 4.6. *Any cofibrant object in* **tame** $([0,\infty), \mathbf{ch})$ *is isomorphic to a direct sum* $\oplus_{i=1}^{l} I^{h_i}\left[b_i,d_i\right)$*, where* l *could possibly be* 0*. Moreover, the decomposition is unique up to isomorphisms.*

Proof. Let X be a cofibrant object in tame$([0,\infty),\mathbf{ch})$, and $0 = t_0 < t_1 < \cdots < t_n$ a sequence in $[0,\infty)$ discretising it. By Proposition 4.1, the morphism $X^{t_{i-1}<t_i}$ is a monomorphism for every $i = 1,\ldots,n$.

Suppose first that all the differentials in X^{t_i} are trivial, for every $i = 0,\ldots,n$. In this case, X is isomorphic to $\oplus_{h\in\mathbb{N}} X_h$. Let $l_h^0 := \dim X_h^0$ and $l_h^{t_i} := \dim\operatorname{coker} X_h^{t_{i-1}<t_i}$ for $i = 1,\ldots,n$. Then X_h is isomorphic to:

$$\bigoplus_{i=0}^{n}\bigoplus_{j=1}^{l_h^{t_i}} I^h\left[t_i,\infty\right)$$

and consequently X is isomorphic to:

$$\bigoplus_{h\in\mathbb{N}}\bigoplus_{i=0}^{n}\bigoplus_{j=1}^{l_h^{t_i}} I^h\left[t_i,\infty\right)$$

and the theorem is proved.

Suppose now that there is a non-trivial differential in X. We can consider the following values:

(i) Let h be the smallest for which $\partial_{h+1}^{t_i}\colon X_{h+1}^{t_i}\to X_h^{t_i}$ is non trivial for some t_i. Note that then $X_h^{t_i} = Z_h X^{t_i}$ for any t.

(ii) Let d be the smallest t_i for which $\partial_{h+1}^{t_i}\colon X_{h+1}^{t_i}\to X_h^{t_i}$ is non trivial.

(iii) Let b be the smallest $t_i \leqslant d$ for which the following intersection contains a non zero element:

$$\operatorname{im}\left(X_h^{t_i\leqslant d}\colon ZX_h^{t_i} = X_h^{t_i}\hookrightarrow X_h^d\right)\cap\operatorname{im}\left(\partial_{h+1}^d\colon X_{h+1}^d\to X_h^d\right)\neq 0$$

We claim that, using these values, it is possible to split the interval sphere $I^h\left[b,d\right)$ out of X and write $X\cong I^h\left[b,d\right)\oplus X'$. If our claim is true, then we can apply the same strategy to X'. If X' has a non-trivial differential, we split out of X' another direct summand of the form $I^{h'}\left[b',d'\right)$. Since the object X is compact, it is guaranteed that this process eventually terminates. At the end of the process, we end up with an object with all the trivial differentials which, by the initial result, can be decomposed as a finite direct sum of interval spheres, and the theorem is proven.

It remain to show our claim: X is isomorphic to a direct sum $I^h\left[b,d\right)\oplus X'$. For that, we make some choices:

1. Choose a non zero vector v in the intersection from step *(iii)* above;
2. Choose x in $X_h^b = Z_h X^d$ and y in X_{n+1}^d such that $X_n^{b\leqslant d}(x) = v = \partial(y)$;
3. Use such x and y to choose a morphism $I(x,y) : I^h\left[b,d\right)\to X$, as described in Remark 4.4.

The reason why we make all these choices is to be able to use Proposition 4.5 to assure that $I(x,y) : I^h[b,d) \to X$ is a cofibration. Consider now the morphism $\varphi\colon X \to I^h[b,b)$ that fits into the following commutative diagram. Note that its existence is guaranteed by axiom A4.

$$\begin{array}{ccc} I^h[b,d) & \xrightarrow{\quad i \quad} & I^h[b,b) \\ {\scriptstyle I(x,y)}\big\downarrow & \overset{\varphi}{\nearrow} & \big\downarrow{\scriptstyle \sim} \\ X & \xrightarrow{\qquad\quad} & 0 \end{array}$$

If $t_i < d$, then, by how we defined d, the differential $\partial^{t_i}_{h+1}\colon X^{t_i}_{h+1} \to X^{t_i}_h$ is trivial. Hence, for any $b \leqslant t_i < d$, the linear transformation $\varphi^{t_i}_{h+1}\colon X^{t_i}_{h+1} \to I^h[b,b)^{t_i}_{h+1}$ has to be trivial. A direct computation, then, shows that $\varphi\colon X \to I^h[b,b)$ factors as:

$$\begin{array}{ccc} & & I^h[b,d) \\ & \overset{\psi}{\nearrow} & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow{\ \varphi\ } & I^h[b,b) \end{array}$$

Therefore, the following composition is the identity:

$$I^h[b,d) \xrightarrow{I(x,y)} X \xrightarrow{\ \psi\ } I^h[b,d), \qquad \psi \circ I(x,y) = \mathbf{1}$$

and consequently X is isomorphic to a direct sum $I^h[b,d) \oplus X'$, proving the claim and thus the decomposition.

Since the interval spheres are indecomposable, by Remark 2.17 such a decomposition is unique. □

As a consequence of Theorem 4.6, we obtain the following characterisation of cofibrant tame parametrised chain complexes with trivial homology.

Corollary 4.7. *A cofibrant object in* $\mathbf{tame}([0,\infty), \mathbf{ch})$ *has trivial homology if and only if it is the direct sum of interval spheres* $I^h[b,b)$*, for various* $h \in \mathbb{N}$ *and* b *in* $[0,\infty)$*.*

In this chapter, we proved the decomposition theorem for cofibrant objects. With it, we can associate for at least this class of tame parametrised chain complexes the invariants provided by the number and type of their indecomposables. Now we aim to extract invariants for any tame parametrised chain complexes by approximating them with cofibrant objects.

Chapter 5

Invariants for tame parametrised objects in model categories

The goal of this chapter is to define invariants for tame parametrised chain complexes using the model structure introduced on $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$. To obtain such invariants, we are going to approximate any tame parametrised chain complex with a cofibrant object. In general, an object in a model category admits many possible cofibrant approximations. Using the concept of minimality, it is possible to choose some cofibrant approximations uniquely. We concentrate on two such approximations, called *minimal cover* and *minimal representative*. Minimal covers and minimal representatives do not need to exist in general in a model category. We show, however, that they both exits in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$.

The chapter is structured as follows. In Section 5.1, we define two notions of minimality. In Section 5.2, we show how to use the existence of the minimal cover in $\mathcal{M}$ to prove the existence of minimal cover in $\mathbf{tame}\left([0,\infty), \mathcal{M}\right)$. In Section 5.3, we study the minimality in the category of compact chain complexes. In Section 5.4, we build and characterise the minimal cover in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$, and we prove the existence of the minimal representative in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$. Finally, in Section 5.5, we study the minimality for the three classes of objects presented in Section 2.5.

Throughout this chapter, the symbol $\mathcal{M}$ denotes a category with a fixed model structure.

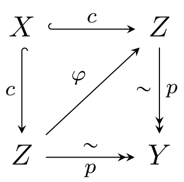
5.1 Minimality in a model category

In model category theory, the idea of minimality is used to uniquely determine some particular morphisms, such as minimal (co)fibrations. Typically, minimality is expressed by the property that any weak equivalence between minimal (co)fibrations is an isomorphism. In general, a model category does not need to admit minimal morphisms.

Thus, the discussion of minimality is usually restricted to specific model categories. See for example [39].

Axiom A5 guarantees the existence of factorisations of morphisms in a model category $\mathcal{M}$, but it does not specify any uniqueness. A morphism may admit many such factorisations. There are model categories in which among all these factorisations there are canonical ones called minimal.

Definition 5.1. Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$. A factorisation $f = p \circ c$, where c is cofibration, and p is a fibration and a weak equivalence, is called *minimal* if every morphism φ which makes the diagram commute is an isomorphism:



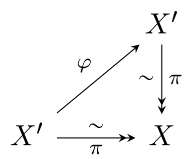
The fibration and weak equivalence in the minimal factorisation of $\emptyset \to X$ is called a *minimal cover* of X.

According to the above definition, we can think about a minimal cover of X as a morphism $\pi\colon X' \to X$ such that:

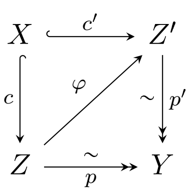
(i) X' is cofibrant,

(ii) π is both a fibration and a weak equivalence

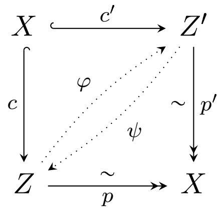
(iii) every morphism φ which makes the following diagram commute is an isomorphism:



Proposition 5.2. *Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$. Assume that $p \circ c = f = p' \circ c'$ are minimal factorisations, through Z and Z', respectively. Then there is an isomorphism φ making the following diagram commute:*



Proof. Let φ and ψ be morphisms making the following diagram commute, which exist by the lifting axiom A4:



Since the diagram commutes, by definition of minimal factorisation both the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are isomorphisms. Consequently, so are φ and ψ. □

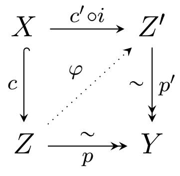
As a direct consequence, we have the following result, showing that minimal covers are invariants in the model categories where they exist.

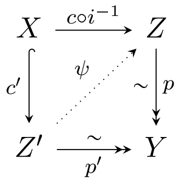
Corollary 5.3. *If a minimal cover exists, then it is unique up to isomorphisms.*

Moreover, the minimal factorisation is an invariant for morphisms.

Proposition 5.4. *Let $f\colon X \to Y$ and $f'\colon X' \to Y$ be two isomorphic morphisms in $\mathcal{M} \downarrow Y$. Then the minimal factorisations of f and f' are isomorphic.*

Proof. Let $X \xhookrightarrow{c} Z \xrightarrow{p} Y$ and $X' \xhookrightarrow{c'} Z' \xrightarrow{p'} Y$ be the minimal factorisations of f and f'. Let $i\colon X \to X'$ be an isomorphism between f and f' in $\mathcal{M} \downarrow Y$. Then the following solid diagrams commute:





The dotted arrows φ and ψ exists by axiom A4. By definition of minimal factorisation, both the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are isomorphisms, thus also φ and ψ are isomorphisms, proving the claim. □

We now introduce another notion of minimality called minimal representative. We first recall:

Definition 5.5. Two objects X and Y in $\mathcal{M}$ are called *weakly equivalent* if there is a finite sequence of weak equivalences of the form:

$$X \xleftarrow{\sim} A_0 \xrightarrow{\sim} A_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} A_n \xrightarrow{\sim} Y$$

Remark 5.6. Being weakly equivalent is an equivalence relation. The identity is a weak equivalence by Proposition 3.4, and thus every object is weakly equivalent to itself. From the definition, it follows that if X is weakly equivalent to Y then also Y is weakly equivalent to X. Finally, the juxtaposition of the sequence connecting X and Y and the sequence connecting Y and Z gives a sequence of weak equivalences connecting X and Z.

Similarly to the classes of factorisations of morphisms, the classes of weakly equivalent objects are large. There are model categories, however, where these classes contain a canonical object called a minimal representative.

Definition 5.7. An object X in $\mathcal{M}$ is called *minimal* if it is cofibrant, fibrant, and every weak equivalence $X \xrightarrow{\sim} X$ is an isomorphism.

Definition 5.8. Let X and Y be two object in $\mathcal{M}$. Y is called *minimal representative* of X if it is minimal and weakly equivalent to X.

The isomorphism type of minimal representatives depends on the homotopy type.

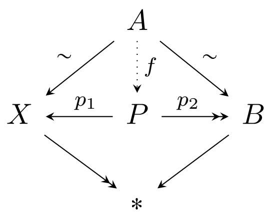
Proposition 5.9. *Let X and X' be weakly equivalent objects in $\mathcal{M}$. If Y is a minimal representative of X and Y' is a minimal representative of X', then Y and Y' are isomorphic.*

To prove this proposition, we use two lemmas.

Lemma 5.10. *Let X be a fibrant object and let $X \xleftarrow{\sim} A \xrightarrow{\sim} B$ a sequence of weak equivalences in $\mathcal{M}$. Then there exists an object C in $\mathcal{M}$ with a weak equivalence and a fibration which is also a weak equivalence that fits in the following diagram:*

$$X \xleftarrow{\sim} C \xrightarrow{\sim} B$$

Proof. Complete the diagram $X \xleftarrow{\sim} A \xrightarrow{\sim} B$ to a commutative square with the terminal object. Since X is fibrant, the morphism $X \to *$ is a fibration. Take now the pullback of $X \twoheadrightarrow * \longleftarrow B$. Then, drawing all these constructions, we have the following diagram:



where the morphism f is induced by the universal property of pullback. The morphism p_2 is a fibration by Proposition 3.5. By axiom A5.(F2), f factorises through an object C as $p \circ c$, where c is a cofibration and a weak equivalence, and p is a fibration. By axiom A2, the compositions $p_1 \circ p$ and $p_2 \circ p$ are weak equivalences. Moreover, by Proposition 3.6, the composition $p_2 \circ p$ is also a fibration. Thus, we have the claimed morphisms: $X \xleftarrow[\sim]{p_1 \circ p} C \xrightarrow[\sim]{p_2 \circ p} B$. □

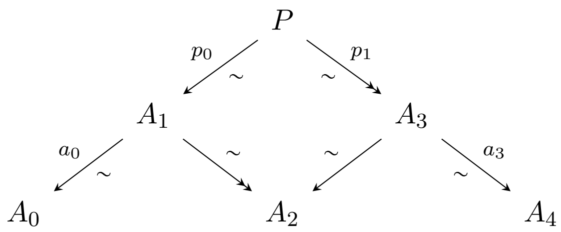
Lemma 5.11. *Consider the following sequence of weak equivalences, where the second morphism is also a fibration:*

$$A_0 \xleftarrow{\sim} A_1 \xrightarrow{\sim} A_2 \xleftarrow{\sim} A_3 \xrightarrow{\sim} A_4$$

Then there exists an object P and two weak equivalences in the form:

$$A_0 \xleftarrow{\sim} P \xrightarrow{\sim} A_4$$

Proof. Compute the pullback of $A_1 \xrightarrow{\sim} A_2 \xleftarrow{\sim} A_3$, and consider the resulting diagram:



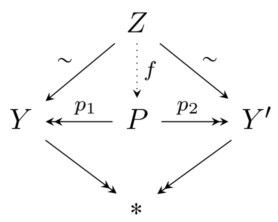
where, by Proposition 3.5, the morphism p_1 is a fibration and a weak equivalence. By axiom A2, the morphism $a_3 \circ p_1$, the morphism p_0, and hence the morphism $a_0 \circ p_0$ are weak equivalences. Thus, we obtain the following diagram, proving the claim: $A_0 \xleftarrow[\sim]{a_0 \circ p_0} P \xrightarrow[\sim]{a_0 \circ p_0} A_4$. □

We are now ready to prove Proposition 5.9.

Proof of Proposition 5.9. Let Y and Y' be minimal representative of weakly equivalent objects X and X'. Since they are weakly equivalent, there exists a sequence of weak equivalences connecting them:

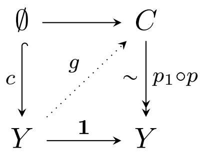
$$X \xleftarrow{\sim} A_0 \xrightarrow{\sim} A_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} A_n \xrightarrow{\sim} Y$$

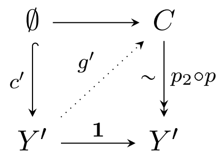
Note that n is even. By applying repeatetly $\frac{n}{2}$ times Lemma 5.10 and Lemma 5.11, we obtain $Y \xleftarrow{\sim} Z \xrightarrow{\sim} Y'$. Complete this sequence to a commutative square with the terminal object. Since Y and Y' are fibrant, the morphisms $Y \to *$ and $Y' \to *$ are fibrations. Consider then the pullback of $Y \twoheadrightarrow * \twoheadleftarrow Y'$. We depict all the constructions in the following diagram:



where the morphism f is the mediating morphism of pullback. The morphisms p_1 and p_2 are fibrations by Proposition 3.5. By axiom A5.(F2), f factorises through an object C as $p \circ c$, where c is a cofibration and a weak equivalence, and p is a fibration.

By axiom A2, the composition $p_1 \circ p$ and $p_2 \circ p$ are weak equivalences. Moreover, by Proposition 3.6, they are also fibrations. Thus, by axiom A4.(S2), the dotted arrows in the following diagrams exist.





where the morphisms c and c' are cofibrations because Y and Y' are cofibrant. By Proposition 3.4 and axiom A2, both g and g' are weak equivalences. Then we have two morphisms $\varphi = p_2 \circ p \circ g\colon Y \to Y'$ and $\psi = p_1 \circ p \circ g'\colon Y' \to Y$. By axiom A2, φ and ψ are weak equivalences. Since both Y and Y' are minimal, the compositions $\varphi \circ \psi\colon Y \to Y$ and $\psi \circ \varphi\colon Y' \to Y'$ are isomorphisms. Consequently, so are φ and ψ, and the claim is proved. □

Corollary 5.12. *The minimal representatives of weakly equivalent objects in $\mathcal{M}$ are isomorphic.*

In particular, if two objects are isomorphic, then their minimal representatives are isomorphic.

Proposition 5.2 and Proposition 5.9 ensure the uniqueness of minimal factorisations, minimal covers and minimal representatives up to isomorphisms. However, they do not imply the existence of any of them. Existence has to be proven separately, and it depends on the considered model category.

Definition 5.13. A model category satisfies the *minimal factorisation axiom* if minimal factorisations exist in the category. A model category satisfies the *minimal representative axiom* if minimal representatives exist in the category.

5.2 Minimal factorisation in tame($[0, \infty)$, $\mathcal{M}$)

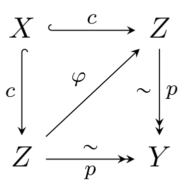
We now explain how to build minimal factorisation in $\mathbf{tame}\left([0,\infty), \mathcal{M}\right)$, provided that $\mathcal{M}$ satisfies the minimal factorisation axiom. Let $f\colon X \to Y$ be a morphism in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$, and $0 = t_0 < \cdots < t_n$ a sequence discretising X and Y. Consider the factorisation built in Construction 3.17, with the following choices:

(i) at step 0, take the minimal factorisation of f_0;

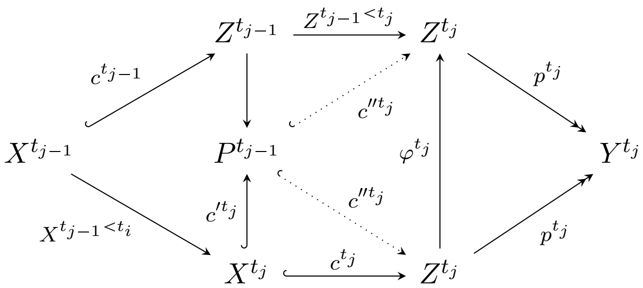
(ii) at step t_j, take the minimal factorisation of $\hat{f}_{t_j}$.

Proposition 5.14. *Assume that $\mathcal{M}$ satisfies the minimal factorisation axiom. Then steps (i) and (ii) give a minimal factorisation in* $\mathbf{tame}\left([0,\infty), \mathcal{M}\right)$.

Proof. The choises *(i)* and *(ii)* in Construction 3.17 give a factorisation $f = p \circ c$, where c is a cofibration and p is a fibration and a weak equivalence. It is left to show that such a factorisation is minimal. Consider φ in the following commutative diagram:



Let $0 = t_0 < \cdots < t_n$ be a common discretising sequence for X, Y and Z. We prove that φ^t is an isomorphism for each t in such a sequence. Since $\mathcal{M}$ satisfies the minimal factorisation axiom, the claim is true for φ^0. Consider now any $j \in \{1, \ldots, n\}$. Draw the following diagram:



where the left side is a pushout, and the morphisms c^{t_j} and p^{t_j} are given by Construction 3.17. Since both $Z^{t_{j-1}}$ and X^{t_j} maps to Z^{t_j}, by universal property of the pushout there exists a morphism $P^{t_{j-1}} \to Z^{t_j}$. By uniqueness of the mediating morphism, such a morphism is precisely the morphism c''^{t_j} of Construction 3.17, and it holds $c''^{t_j} = \varphi^{t_j} \circ c''^{t_j}$. Hence, the middle inner diagram commutes. Then by minimality of the factorisation of $\hat{f}^{t_j}$ in $\mathcal{M}$, φ^{t_j} is an isomorphism, proving the claim. □

Note that we need to know the explicit minimal factorisation of morphisms in $\mathcal{M}$ to apply this construction and build the minimal factorisation in $\mathbf{tame}\left([0,\infty), \mathcal{M}\right)$.

Corollary 5.15. *If $\mathcal{M}$ satisfies the minimal factorisation axiom, then every object in* $\mathbf{tame}\left([0,\infty), \mathcal{M}\right)$ *admits a minimal cover.*

In particular, if $\mathcal{M}$ satisfies the minimal factorisation axiom, the minimal cover of any cofibrant object X in $\mathbf{tame}\left([0,\infty), \mathcal{M}\right)$ is $\mathbf{1}\colon X \to X$.

5.3 Minimality in ch

Our next task is to present an explicit construction for the minimal representative and minimal factorisation in **ch**. Recall that all objects in **ch** are fibrant and cofibrant. It follows that the minimal cover of any object X in **ch** is $\mathbf{1}\colon X \to X$.

Minimal representative

Let V be a chain complex with trivial differentials. In this case, $HV = V$ and hence every weak equivalence $\varphi\colon V \to V$ is an isomorphism. More generally, we have the following result.

Theorem 5.16. *An arbitrary chain complex X is minimal if and only if all its differentials are trivial.*

Proof. If all the differentials are trivial, then $X = HX$. Hence, by Definition 3.9, it follows that every weak equivalence $\varphi\colon X \to X$ is an isomorphism. On the other hand, suppose X is minimal and it has at least one nontrivial differential in some degree h. Then, considering the decomposition (3.2) of X, at least one direct summand is a disk: $X \cong \bigoplus_{i=0}^{l} D^h \oplus \bigoplus_{h'} S^{h'}$, with $l > 0$. Define $\varphi\colon X \to X$ as the projection onto $\bigoplus_{h'} S^{h'}$. The morphism φ is a weak equivalence since $\bigoplus_{i=0}^{l} D^h$ has trivial homology, but it is not an isomorphism. This is a contradiction since X is minimal. $\square$

We can now use this result to describe the minimal representatives in **ch**. It follows that **ch** satisfies the minimal representative axiom.

Proposition 5.17. *The minimal representative of an object X in* **ch** *is HX.*

Proof. By definition, HX has all trivial differentials, and thus, by Theorem 5.16 is minimal. Recall the morphism $s\colon HX \to X$ defined in Section 3.2. Such a morphism is in particular a weak equivalence, and thus HX is the minimal representative of X. $\square$

Theorem 5.16 can be generalised to the following proposition.

Proposition 5.18. *Let $f\colon X \hookrightarrow Y$ be a cofibration in* **ch** *such that the chain complex $Y/f(X)$ has all trivial differentials. Then f satisfies the following minimality condition: every weak equivalence $\varphi\colon Y \to Y$ for which $\varphi \circ f = f$ is an isomorphism.*

Proof. Consider the following solid diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \overset{f}{\hookrightarrow} & Y & \twoheadrightarrow & Y/f(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbf{1}} & & {\scriptstyle\sim}\downarrow{\scriptstyle \varphi} & & \vdots & & \\ 0 & \longrightarrow & X & \overset{f}{\hookrightarrow} & Y & \twoheadrightarrow & Y/f(X) & \longrightarrow & 0 \end{array} \tag{5.3.1}$$

The dotted arrow is given by Proposition 1.32 and makes the diagram commute. Theorem 1.39 applied to (5.3.1) gives the following long exact sequences in homology:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_{h+1}X & \longrightarrow & H_{h+1}Y & \longrightarrow & H_{h+1}Y/f(X) & \longrightarrow & H_hX & \longrightarrow & H_hY & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle \cong} & & \\ \cdots & \longrightarrow & H_{h+1}X & \longrightarrow & H_{h+1}Y & \longrightarrow & H_{h+1}Y/f(X) & \longrightarrow & H_hX & \longrightarrow & H_hY & \longrightarrow & \cdots \end{array}$$

By the Five Lemma 1.29, the central morphism in the above diagram is an isomorphism. Thus, the morphism $Y/f(X) \to Y/f(X)$ is a weak equivalence. Since $Y/f(X)$ has all trivial differentials, by Theorem 5.16 it follows that $Y/f(X) \to Y/f(X)$ is an isomorphism. Then, by applying Lemma 1.30 to the diagram (5.3.1), we have that φ is an isomorphism. □

Minimal factorisation

The goal of this subsection is to build the minimal factorisation of a chain map.

Construction 5.19. Let $f\colon X \to Y$ be a morphism of chain complexes. To construct its minimal factorisation we perform the following steps:

1. Take the kernel $\kappa\colon K \hookrightarrow X$ of f;
2. Choose an isomorphism $K \xrightarrow{\cong} DBK \oplus HK$, which exists because of the standard decomposition of chain complexes (Section 3.2);
3. Consider the composition:

$$\alpha : K \xrightarrow{\cong} DBK \oplus HK \xhookrightarrow{\begin{bmatrix}\mathbf{1} & 0 \\ 0 & i\end{bmatrix}} DBK \oplus DHK$$

   where i is defined in Construction 1.35. Note that α is a cofibration, since it is the composition of cofibrations;

4. use axiom A4 to construct a morphism $\varphi\colon X \to DBK \oplus DHK$ which fits into the following commutative diagram:

$$\begin{array}{ccc} K & \overset{\alpha}{\hookrightarrow} & DBK \oplus DHK \\ {\scriptstyle\kappa}\big\downarrow & \overset{\varphi}{\nearrow} & \big\downarrow{\scriptstyle\sim} \\ X & \longrightarrow & 0 \end{array}$$

Note that the morphism $\begin{bmatrix}\varphi \\ f\end{bmatrix} : X \to (DBK \oplus DHK) \oplus Y$ is a cofibration. ■

Proposition 5.20. *The following factorisation is minimal:*

$$\begin{array}{ccccc} & & (DBK \oplus DHK) \oplus Y & & \\ & \overset{\begin{bmatrix}\varphi \\ f\end{bmatrix}}{\nearrow} & & \overset{\begin{bmatrix}0 & \mathbf{1}\end{bmatrix}}{\searrow}{\scriptstyle\sim} & \\ X & & \xrightarrow{\qquad f \qquad} & & Y \end{array}$$

Proof. The morphism $\begin{bmatrix}\varphi \\ f\end{bmatrix}$ is a cofibration, and the morphism $\begin{bmatrix}0 & \mathbf{1}\end{bmatrix}$ is a fibration and a weak equivalence, since $DBK \oplus DHK$ has trivial homology. We need to show that the factorisation is minimal. Consider a morphism

$$\psi = \begin{bmatrix}\psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22}\end{bmatrix} \colon (DBK \oplus DHK) \oplus Y \to (DBK \oplus DHK) \oplus Y$$

such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "{\begin{bmatrix}\varphi \\ f\end{bmatrix}}"] \arrow[d, hook, "{\begin{bmatrix}\varphi \\ f\end{bmatrix}}"'] & (DBK \oplus DHK) \oplus Y \arrow[d, two heads, "\sim"', "{\begin{bmatrix}0 & \mathbf{1}\end{bmatrix}}"] \\ (DBK \oplus DHK) \oplus Y \arrow[ur, "\psi"] \arrow[r, two heads, "\sim", "{\begin{bmatrix}0 & \mathbf{1}\end{bmatrix}}"'] & Y \end{tikzcd}$$

Commutativity of the bottom triangle implies that $\psi_{21} = 0$ and $\psi_{22} = \mathbf{1}$. Since K is the kernel of f, commutativity of the top triangle implies commutativity of

$$\begin{tikzcd} K \arrow[r, hook, "\alpha"] \arrow[d, hook, "\varphi\circ\kappa=\alpha"'] & DBK \oplus DHK \\ DBK \oplus DHK \arrow[ur, "\psi_{11}"] & \end{tikzcd}$$

A direct computation shows that the quotient $(DBK \oplus DHK)/\alpha(K)$ is SHK, which has all trivial differentials. Thus, the morphism $\psi_{11} \colon DBK \oplus DHK \to DBK \oplus DHK$ is an isomorphism by Proposition 5.18. It follows that also $\psi = \begin{bmatrix}\psi_{11} & \psi_{12} \\ 0 & \mathbf{1}\end{bmatrix}$ is an isomorphism, proving the claim. □

5.4 Minimality in tame([0, ∞), ch)

In this section, we present an explicit construction of minimal covers in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$. Moreover, we provide a characterisation for them, and we prove that $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$ satisfies the minimal representatives axiom.

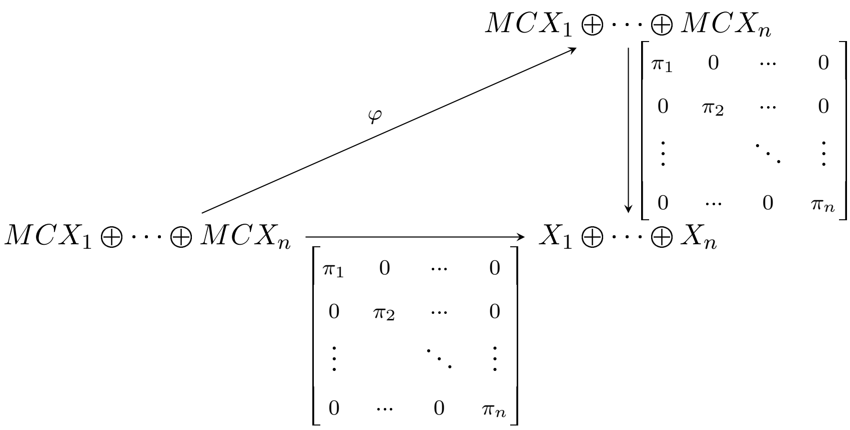
Note that every object in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$ is fibrant. Recall that, by Proposition 4.1, an object is cofibrant in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$ if and only if its transition morphisms are monomorphisms for every $s < t$ in $[0,\infty)$.

Minimal cover

By Proposition 5.14, we know that the minimal cover exists in $\mathbf{tame}\left([0,\infty), \mathbf{ch}\right)$. We illustrate here some of its properties and how to construct it.

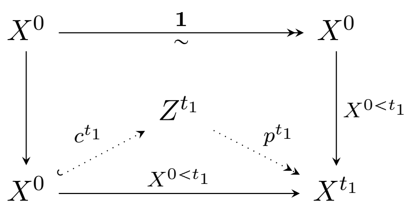
Proposition 5.21. *In* **tame** $([0,\infty), \mathbf{ch})$*, the minimal cover preserves direct sums.*

Proof. Let $X_1, \ldots, X_n$ be objects in **tame** $([0,\infty), \mathbf{ch})$, and denote by X their direct sum $X = X_1 \oplus \cdots \oplus X_n$. Since a minimal cover is unique up to isomorphism, it is enough to show that $\pi\colon MCX_1 \oplus \cdots \oplus MCX_n \to X$ is a minimal cover of X, where the map π is given by the direct sum of the minimal covers $\pi_i\colon MCX_i \to X_i$. $MCX_1 \oplus \cdots \oplus MCX_n$ is cofibrant being the direct sum of cofibrant objects. π is a fibration because it is the direct sum of epimorphisms, and a weak equivalence since homology preserves direct sums. It is left to prove that any morphism φ in the following diagram is an isomorphism.



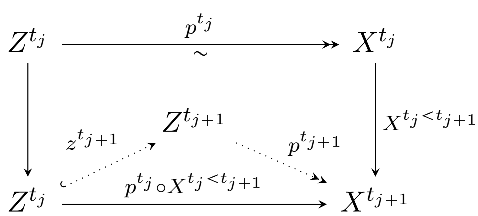
This follows by commutativity and minimality condition on π_i, for all $i = 1, \ldots, n$. □

We describe now the steps to build the construction of the minimal cover of an object X in **tame** $([0,\infty), \mathbf{ch})$. Let us consider a discretising sequence $0 = t_0 < \cdots < t_n$ for X. At step 0, the minimal factorisation of $0 \to X^0$ is $\mathbf{1}\colon X^0 \to X^0$. According to Proposition 5.14, we build the following diagram:



(5.4.1)

The morphisms c^{t_1} and p^{t_1} are given by the minimal factorsation of $X^{0<t_1}$ (Construction 5.19). By Proposition 5.14, we can iterate the process for every t_i, $i = 0, \ldots, n$. In particular, at step t_j the diagram (5.4.1) takes the form:



Note that the construction is heavily based on the minimal factorisation in **ch**.

We now present a characterisation of minimal covers in $\mathbf{tame}([0,\infty),\mathbf{ch})$. Before presenting the characterisation, we prove the following lemma.

Lemma 5.22. *Let X be in $\mathbf{tame}([0,\infty),\mathbf{ch})$. Let $\varphi\colon X \to X$ be an endomorphism. Then the decreasing sequence $\operatorname{im}(\varphi) \supseteq \operatorname{im}(\varphi^2) \supseteq \cdots$, where φ^i is the i-th composition of φ with itself, stabilises.*

Proof. Since X is an object in $\mathbf{tame}([0,\infty),\mathbf{ch})$, there exists a sequence $0 = t_0 < t_1 < \cdots < t_n$ discretising it. We claim that the sequence discretises also $\operatorname{im}\varphi^i$, for all $i \in \mathbb{N}$. To see it, consider $s < t \in [t_j, t_{j+1})$, for all $j = 0,\dots,n$. Since the transition morphism $X^{s<t}$ is an isomorphism, also the morphism $\operatorname{im}(\varphi^s)^i \to \operatorname{im}(\varphi^t)^i$ is so, and the claim is proved.

We then need to verifies that the sequence $\operatorname{im}(\varphi) \supseteq \operatorname{im}(\varphi^2) \supseteq \cdots$ stabilise on each t_j, $j = 0,\dots,n$. This follows from the compactness of the objects in **ch** (Remark 2.10). □

Proposition 5.23. *Let X be in $\mathbf{tame}([0,\infty),\mathbf{ch})$. Let Y be a cofibrant object in $\mathbf{tame}([0,\infty),\mathbf{ch})$, and $f\colon Y \xrightarrow{\sim} X$ a fibration and a weak equivalence. Then f is a minimal cover if and only if no direct summand $I^h[b,b)$ of Y is mapped to zero under f.*

Proof. Assume f is the minimal cover of X, and suppose there exists a direct summand $I^h[b,b)$ of Y such that $f\left(I^h[b,b)\right) = 0$. Decompose Y as $Y \cong Y' \oplus I^h[b,b)$. Define the endomorphism $\varphi\colon Y' \oplus I^h[b,b) \to Y' \oplus I^h[b,b)$ as

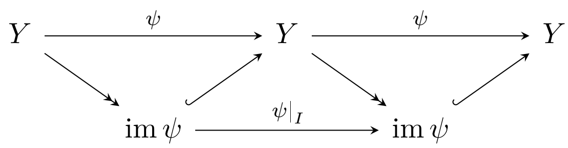
$$\varphi := \left[\begin{array}{c|c} \mathbf{1} & 0 \\ \hline 0 & 0 \end{array}\right]$$

Since $f\left(I^h[b,b)\right) = 0$, the following diagram commutes:

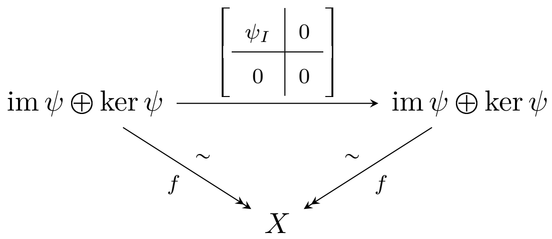
$$\begin{array}{ccc} Y' \oplus I^h[b,b) & \xrightarrow{\quad\varphi\quad} & \overline{Y} \oplus I^h[b,b) \\ & {\scriptstyle f,\,\sim}\searrow \quad \swarrow{\scriptstyle f,\,\sim} & \\ & X & \end{array}$$

Since φ is not an isomorphism, $f\colon Y \xrightarrow{\sim} X$ is not minimal.

Assume now that Y does not contain any summand $I^h[b,b)$, for b in $[0,\infty)$. We need to show that f is minimal, i.e. that every endomorphism $\varphi\colon Y \to Y$ such that $f = f \circ \varphi$ is an isomorphism. By Lemma 5.22, there exists m such that $\operatorname{im}\varphi^m = \operatorname{im}\varphi^{m'}$ for all $m' > m$. Define $\psi := \varphi^m$. By compactness, to prove that φ is an isomorphism is equivalent to prove that ψ is a monomorphism. Since $\operatorname{im}\psi = \operatorname{im}\psi^2$, the map $\psi|_I$ in the following commutative diagram is an isomorphism:



Thus, $\operatorname{im}\psi$ is a direct summand of Y. We can write:



Note that $\ker\psi$ is still a cofibrant object in $\mathbf{tame}([0,\infty),\mathbf{ch})$. Moreover, since f is a weak equivalence and $f = f \circ \psi$, $\ker\psi$ has trivial homology. By Corollary 4.7, $\ker\psi$ is a sum of interval spheres $I^h[b,b)$. From $f = f \circ \psi$, it follows that $\ker\psi \subseteq \ker f$. This means that, if $\ker\psi$ is not trivial, there is some direct summand $I^h[b,b)$ sent to zero under f. This is a contradiction, hence ψ is a monomorphism, proving the claim. □

Minimal representative

In this section, we prove the existence of minimal representatives in $\mathbf{tame}([0,\infty),\mathbf{ch})$.

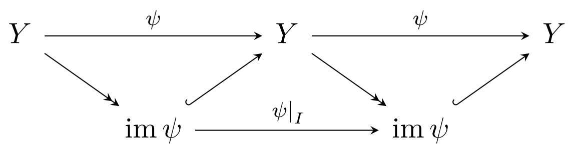
Proposition 5.24. *Let X and Y be objects in $\mathbf{tame}([0,\infty),\mathbf{ch})$, such that Y is fibrant, cofibrant and weakly equivalent to X. Let $\oplus_{b,d} I^h[b,d)$ be the decomposition of Y according to Theorem 4.6. Y is the minimal representative of X if and only if it holds $b < d$, for all $I^h[b,d)$ in $\oplus_{b,d} I^h[b,d)$.*

Proof. Suppose that Y is the minimal representative of X and has a direct summand isomorphic to $I^h[b,b)$, for some b in $[0,\infty)$ and $h \in \mathbb{N}$. Decompose Y as $Y \cong Y' \oplus I^h[b,b)$. Define the endomorphism $\varphi\colon Y' \oplus I^h[b,b) \to Y' \oplus I^h[b,b)$ as

$$\varphi := \left[\begin{array}{c|c} \mathbf{1} & 0 \\ \hline 0 & 0 \end{array}\right]$$

Since $I^h[b,b)$ has trivial homology, φ is a weak equivalence, but it is not an isomorphism. Hence, Y is not minimal, which is a contradiction.

Assume now $b < d$, for all $I^h[b,d)$ in the decomposition of Y. We prove that Y is minimal, i.e. that every weak equivalence $f\colon Y \to Y$ is an isomorphism. By Lemma 5.22, there exists m such that $\operatorname{im} f^m = \operatorname{im} f^{m'}$ for all $m' > m$. Define $\psi := f^m$. By axiom A2, ψ is a weak equivalence. By compactness, to prove that f is an isomorphism is equivalent to prove that ψ is a monomorphism. Since $\operatorname{im}\psi = \operatorname{im}\psi^2$, the map $\psi|_I$ in the following commutative diagram is an isomorphism:



Thus, $\operatorname{im}\psi$ is a direct summand of Y. We can write:

$$\operatorname{im}\psi \oplus \ker\psi \xrightarrow{\left[\begin{array}{c|c} \psi_I & 0 \\ \hline 0 & 0 \end{array}\right]} \operatorname{im}\psi \oplus \ker\psi$$

Note that $\ker\psi$ is a cofibrant object in $\mathbf{tame}([0,\infty),\mathbf{ch})$. Moreover, since ψ is a weak equivalence, $\ker\psi$ has trivial homology. By Corollary 4.7, $\ker\psi$ is a sum of interval spheres $I^h[b,b)$. This is a contradiction. Then $\ker\psi$ is trivial and ψ is a monomorphism, proving the claim. □

This characterisation provides a method to construct the minimal representative of an object X of $\mathbf{tame}([0,\infty),\mathbf{ch})$, using the following two steps:

(i) take the cofibrant replacement $f\colon Y \to X$ of X;

(ii) decompose Y into interval spheres by Theorem 4.6, and retain only the summands $I^h[b,d)$ such that $b < d$.

Proposition 5.25. *Steps (i) and (ii) give minimal representatives in* $\mathbf{tame}([0,\infty),\mathbf{ch})$.

Proof. Let X be an object in $\mathbf{tame}([0,\infty),\mathbf{ch})$. Perform step *(i)* and *(ii)* on it and obtain a cofibrant object Y, weakly equivalent to X. Recall that all objects in $\mathbf{tame}([0,\infty),\mathbf{ch})$ are fibrant. Note that, since the interval sphere $I^h[b,b)$ are contractible, for every $h \in \mathbb{N}$ and b in $[0,\infty)$, step *(ii)* does not change the homology. By Proposition 5.24, f is the minimal cover of X. □

Corollary 5.26. *The category* $\mathbf{tame}([0,\infty),\mathbf{ch})$ *satisfies the minimal representative axiom.*

In particular, in $\mathbf{tame}([0,\infty),\mathbf{ch})$ the minimal representative of an object X can be obtained splitting out of the summands $I^h[b,b)$, for some b in $[0,\infty)$, from the minimal cover of X.

5.5 Minimality in the motivational examples

Minimality for parametrised vector spaces

Let V be a parametrised vector space. Since V is a tame parametrised chain complex concentrated in degree 0 in $[0,\infty)$, it follows that its minimal cover cannot contain any $I^h[b,b)$ summand. Since, by Proposition 5.24, the minimal representative of an object in $\mathbf{tame}([0,\infty),\mathbf{ch})$ can be obtained by splitting out the direct summands $I^h[b,b)$ from its minimal cover, we have the following proposition:

Proposition 5.27. *For parametrised vector spaces, the minimal cover and the minimal representative are isomorphic.*

Let $b < d$ be in $[0, \infty)$. The minimal cover of $\mathbb{I}_{[b,d)}$ is $I^0\,[b, d)$. Recall that $H_0\left(I^0\,[b, d)\right) = \mathbb{I}_{[b,d)}$. It follows

Theorem 5.28. *Minimal covers and minimal representatives are complete invariants for parametrised vector spaces.*

We present an example. Let V be the parametrised vector space shown in the following diagram.

$$\begin{array}{ccccc} 0 & & t_1 & & t_2 \\ \mathtt{k}^2 & \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} & \mathtt{k} & \longrightarrow & 0 \end{array}$$

Its minimal cover, and thus also its minimal representative, is depicted in the following diagram.

$$\begin{array}{ccccccc} 0 & & t_1 & & t_2 & & \\ 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} & \mathtt{k}^2 & \xrightarrow{\mathbf{1}} & \cdots \\ \downarrow & & \downarrow{\scriptstyle\begin{pmatrix}1 & 0\end{pmatrix}} & & \downarrow{\scriptstyle\mathbf{1}} & & \\ \mathtt{k}^2 & \xrightarrow{\mathbf{1}} & \mathtt{k}^2 & \xrightarrow{\mathbf{1}} & \mathtt{k}^2 & \xrightarrow{\mathbf{1}} & \cdots \end{array}$$

We obtain the decomposition $I^0\,[0, t_1) \oplus I^0\,[0, t_2)$ of the minimal cover. By Theorem 2.13, V decomposes as $\mathbb{I}_{[0,t_1)} \oplus \mathbb{I}_{[0,t_2)}$.

Minimality for tame commutative ladders

As opposed to what happens for tame parametrised vector spaces, neither minimal covers nor minimal representatives are complete invariants for tame commutative ladders. As an example, let X and Y be two commutative ladders described graphically as:

$$X := \begin{array}{ccccccccccc} \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\mathbf{1}} & & \downarrow{\scriptstyle\mathbf{1}} & & \downarrow & & \downarrow & & \downarrow & & \\ \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \end{array}$$

$$\begin{array}{ccccccccc} 0 & & t_1 & & t_2 & & t_3 & & t_4 \end{array}$$

$$Y := \begin{array}{ccccccccccc} \mathtt{k} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\mathbf{1}} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathtt{k} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \end{array}$$

Both their minimal covers are:

$$
\begin{array}{ccccccccccc}
0 & & t_1 & & t_2 & & t_3 & & t_4 & & \\
\mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \\
\downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle \binom{1}{0}} & & \\
\mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\binom{1}{0}} & \mathtt{k}^2 & \xrightarrow{\mathbf{1}} & \mathtt{k}^2 & \xrightarrow{\mathbf{1}} & \cdots
\end{array}
$$

And both their minimal representatives are:

$$
\begin{array}{ccccccccccc}
0 & & t_1 & & t_2 & & t_3 & & t_4 & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots
\end{array}
$$

Although not complete, both the minimal cover and the minimal representative are invariants for commutative ladders. Thus, we can associate to X and Y either the decomposition of their minimal cover, given by $I^0\,[0,0)\oplus I^0\,[t_3,\infty)$, or the decomposition of their minimal representative, given by $I^0\,[t_3,\infty)$. Note that the difference between the two is in the contractible summand $I^0\,[0,0)$.

Minimality for zigzags

We describe now a strategy to retrieve minimal covers and minimal representatives of zigzags. Such a strategy is based on the decomposition Theorem 2.20 and thus does not provide an efficient implementation. We present it to show a theoretical method to retrieve these invariants. We defer to future investigations the study of a constructive way to compute minimal covers and minimal representatives of zigzags.

To retrieve the minimal cover of an object X in **ZigZag**, decompose X into the direct sum of interval zigzags using Corollary 2.24, and compute the minimal covers of each interval zigzag sequence. Since, by Proposition 5.21, the minimal cover preserves direct sums, this strategy provides the minimal cover of X. Moreover, to prove the completeness of the minimal cover in **ZigZag**, it is enough to verify it for interval zigzags. Note that, to be able to apply this strategy, it is necessary to fix the type $\mathcal{C}$.

Once the minimal cover of a zigzag is built, we can apply Proposition 5.25 to it and retrieve the minimal representative of the zigzag. However, since the minimal representative does not preserve direct sums, showing that the minimal representative is a complete invariant for interval zigzags is not enough to prove that the minimal representative is a complete invariant in **ZigZag**.

Using Construction 2.21 and Construction 2.23, we translate zigzag sequences into discrete chain zigzags. There is not a natural way to obtain this. We present here other possibilities, highlighting their drawbacks and thus motivating our original choice.

Counterexample 5.29. Consider the following concatenation of a functor $X\colon [n] \to \mathbf{ch}$ and a directed linear transformation (f, c):

- Assume the chain complex X^n is concentrated in degree g:
  - If $c = r$ and the domain of f coincide with X^n_h, then $X * (f, c) : [n+1] \to \mathbf{ch}$ is given by the sequence of $n + 1$ chain maps:

    $$\xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{S^h f}$$

    where $S^h f$ is the h-th suspension of f.
  - If $c = l$ and the codomain of f coincide with X^n_h, set X^n_{h+1} equal to the domain of f, and $\partial^{X^n}_{h+1} = f$, so that X^n is concentrated in degrees $h, h + 1$. If $n > 0$, the transition morphism $X^{n-1<n}$ remains unchanged in degree h and it is set to zero otherwise. Note that, in this case, we are redefining the functor $X\colon [n] \to \mathbf{ch}$ by modifying its last chain complex.
- Assume the chain complex X^n is concentrated in degrees $\{h, h+1\}$, $c = r$ and the domain of f coincide with X^n_{h+1}. Define Y to be the chain complex concentrated in degree $h+1$ such that Y_{h+1} is the codomain of f. Let $g\colon X^n \to Y$ be the chain map which in degree $h + 1$ is given by f. Define $X * (f, c) : [n + 1] \to \mathbf{ch}$ to be given by the sequence of $n + 1$ chain maps:

  $$\xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{g}$$

Note that, in this case, according to the direction of (f, c) the concatenation varies in length. Moreover, it is not allowed a zigzag sequence where there exists i such that $c_i = c_{i+1} = l$, because, since there is not guarantee that $f_{i+1} \circ f_i = 0$, we would not obtain a chain complex. Applying Construction 2.23 to this concatenation. Taking the left Kan extension, we get a zigzag from each zigzag sequence.

The interval zigzag sequences $\{(\mathtt{k} \to 0, r), (0 \to 0, l)\}$ and $\{(\mathbf{1}_{\mathtt{k}}, r), (\mathbf{1}_{\mathtt{k}}, l)\}$, along the inclusion $[1] \subset [0, \infty)$: $i \mapsto i$ for $i = 0, 1$, correspond to the interval zigzags:

$$\begin{array}{ccccc} 0 & & 1 & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \qquad\qquad \begin{array}{ccccc} 0 & & 1 & & \\ 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \\ \downarrow & & \downarrow{\scriptstyle \mathbf{1}} & & \\ \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \end{array}$$

Both have $I^0[0, 1)$ as minimal cover and minimal representative, but they are not isomorphic. Note that the zigzag sequences are of the same zigzag profile. Thus, fixing the zigzag profile is not enough for turning the minimal cover or the minimal representative into a complete invariant. ■

Counterexample 5.30. Consider the following concatenation of a functor $X\colon [n] \to \mathbf{ch}$ and a directed linear transformation (f, c):

- Assume the chain complex X^n is concentrated in degree g:
  - If $c = r$ and the domain of f coincide with X^n_k, then $X * (f, c) : [n+1] \to \mathbf{ch}$ is given by the sequence of $n+1$ chain maps:

    $$\xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{S^h f}$$

    where $S^h f$ is the h-th suspension of f.
  - If $c = l$ and the codomain of f coincide with X^n_h, then define a chain complex Y to be concentrated in degrees $\{h, h+1\}$ and such that $Y_h = X^n_h$, Y_{h+1} is the domain of f and $\partial^Y_{h+1} = f$. Define the map $g\colon X^n \to Y$ as the chain map which is the identity in degree h and zero otherwise. Define $X * (f, c) : [n+1] \to \mathbf{ch}$ to be given by:

    $$\begin{array}{c|l} \xrightarrow{g} & \text{if } n = 0 \\ \hline \xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{g} & \text{if } n > 0 \end{array}$$

- Assume the chain complex X^n is concentrated in degrees $\{h, h+1\}$, $c = r$ and the domain of f coincide with X^n_{h+1}. Define Y to be the chain complex concentrated in degree $h+1$ such that Y_{h+1} is the codomain of f. Let $g\colon X^n \to Y$ be the chain map which in degree $h+1$ is given by f. Define $X * (f, c) : [n+1] \to \mathbf{ch}$ to be given by the sequence of $n+1$ chain maps:

  $$\xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{g}$$

Note that, as in Construction 2.21, the result of a concatenation is a functor $X\colon [n+1] \to \mathbf{ch}$, and it is possible to concatenated two directed transformations with direction l. Applying Construction 2.23 to this concatenation, and taking the left Kan extension, we obtain a zigzag from each zigzag sequence.

The interval zigzag sequences $\{(\mathbf{1}_{\mathtt{k}}, r), (\mathtt{k} \to 0, r)\}$ and $\{(\mathbf{1}_{\mathtt{k}}, r), (\mathbf{1}_{\mathtt{k}}, l)\}$, along the inclusion $[2] \subset [0, \infty)\colon i \mapsto i$ for $i = 0, 1, 2$, correspond to the interval zigzags:

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathtt{k} & \longrightarrow & \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \qquad \begin{array}{ccccccc} 0 & & 1 & & 2 & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \\ \downarrow & & \downarrow & & \downarrow \mathbf{1} & & \\ \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \end{array}$$

Both have $I^0\,[0, 2)$ as minimal cover and minimal representative, but they are not isomorphic. ■

Counterexample 5.31. Consider the following concatenation of a functor $X\colon [n] \to \mathbf{ch}$ and a directed linear transformation (f, c). Assume X^n is concentrated in one degree.

- Suppose that $c = r$ and the domain of f coincides with X^n_h. In this case, put $X^n_{h+1} = X^n_h$, and $\partial^{X^n}_{h+1} = \mathbf{1}$, so that X^n is concentrated in degrees $h, h+1$. If $n > 0$, the transition morphism $X^{n-1<n}$ remains unchanged in degree h, and it is set to zero otherwise. Then define Y as the chain complex concentrated in degree $h+1$, where it is equal to the codomain of f, and the map $g\colon X^n \to Y$ as the chain map which is f in degree $h+1$ and zero otherwise. Define $X * (f, c) : [n+1] \to \mathbf{ch}$ to be given by:

$$\begin{array}{c|l} \xrightarrow{g} & \text{if } n = 0 \\ \hline \xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{n-1<n}} \xrightarrow{g} & \text{if } n \geqslant 1 \end{array}$$

- Suppose $c = l$ and the codomain of f coincide with X^n_h. In this case, one may choose any of the concatenation described previously (Construction 2.21, Counterexample 5.29, Counterexample 5.30). According to this choice, one may need to define also the case in which X^n is concentrated in two degrees and $c = r$.

According to the choice for the direction $c = l$, we obtain functor of, possibly, different lengths with different properties, but the following example disproves the completeness of the minimal cover using simply a directed transformation with direction $c = r$, regardless of the behaviour of the directed transformations with $c = l$.

According to this concatenation, the zigzag sequences $\{(\mathbf{1}_{\mathtt{k}}, r)\}$ and $\{(\mathtt{k} \xrightarrow{0} \mathtt{k}, r)\}$, along the inclusion $[1] \subset [0, \infty)$: $i \mapsto i$ for $i = 0, 1$, correspond to the zigzags:

$$\begin{array}{ccccc} 0 & & 1 & & \\ \mathtt{k} & \xrightarrow{\mathbf{1}} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \\ {\scriptstyle\mathbf{1}}\downarrow & & \downarrow & & \\ \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \qquad\qquad \begin{array}{ccccc} 0 & & 1 & & \\ \mathtt{k} & \xrightarrow{0} & \mathtt{k} & \xrightarrow{\mathbf{1}} & \cdots \\ {\scriptstyle\mathbf{1}}\downarrow & & \downarrow & & \\ \mathtt{k} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Both have $I^0\,[0,0) \oplus I^1\,[1,\infty)$ as minimal cover, and $I^1\,[1,\infty)$ as minimal representative, but they are not isomorphic. Note that also in this case the zigzag sequences are of the same zigzag profile, and thus it is not enough to fix the zigzag profile for turning the minimal cover or the minimal representative into a complete invariant. ■

Conclusions

In this thesis, we have described a novel approach for the study of topological invariants of data, showing that, using model category theory, it is possible to retrieve homological and homotopical invariants from the simplicial complex modelling a point cloud. We proved that this approach encloses many classes of objects that have demonstrated to be engaging in TDA. Moreover, we showed that the retrieved invariants are in perfect accordance with the complete invariants of persistent homology. These results are exciting and open the way for new investigations. We list here some of the open problems that this approach prepared.

Completeness of the invariants

In Section 2.5, we described a strategy to retrieve a complete invariant for zigzags, namely the minimal cover. Since zigzags already admit a complete invariant in persistent homology theory, it is important to have a complete invariant also in our novel approach. However, such a strategy has the drawback of relying on the structure theorem of zigzag sequences. This means that it has a theoretical relevance, but it does not produce a constructive way of retrieving the invariants. In particular, it does not provide an efficient algorithm, leaving thus open the computational problem. Therefore, we need an alternative way of proving the completeness of the minimal cover of zigzags, hopefully leading to efficient computation.

Stability

A crucial aspect that we did not address in this work is the stability. We described a process that assigns to a set of points in a metric space the indecomposables of the cofibrant approximations of the tame parametrised chain complex induced by the points. For applying this process to real data, we need it to be stable. We have some preliminary results on one of the passages, namely the assignment of the indecomposables of the cofibrant objects. However, these results depend on the chosen metrics, and we would prefer to use a different strategy, not to be forced to pick the distances a priori. In particular, we would like to adopt a technique similar to the one used in [20, 37]. In these works, the idea is to define discrete invariants and to stabilise them using a so-called hierarchical stabilisation, instead of proving some stability results for different metrics. The reason for this choice is that information inside different datasets is summarised at best by different invariants. Thus, instead of choosing an invariant a priori and finding the best distances that stabilise it, it is more convenient to have a general method to ensure stability for any discrete invariant. Using this strategy, we can cherrypick the most meaningful invariant for each point cloud, tailoring the analysis of the data, still ensuring the stability of the result.

Implementation

Another crucial aspect is the implementation of the results we found. Recall the original workflow of this thesis: start with data, build the chain complex of a simplicial complex modelling the data, associate to it its minimal cover (resp. representative), decompose the minimal cover (resp. representative), and use its indecomposables to define invariants. The next goal would be to prove such a workflow to be computable. Some passage has already been implemented. In particular, the decomposition of cofibrant objects can be achieved by previously existing software, for example [6, 7]. Moreover, we remark that we have implemented an algorithm for the decomposition of cofibrant objects, that we aim to release soon. What is left is the computation of minimal covers and minimal representatives of simplicial complexes which are not cofibrant. To prove the workflow to be computable, one could start studying the implementation of this last passage, constructing the minimal cover (resp. representative). Another possibility is to consider the whole workflow as a single step and compute the minimal cover (resp. minimal representative) of a simplicial complex directly as the direct sum of interval spheres. In this case, one cannot rely on the existing software, but one could benefit from a more direct, and, hopefully, more efficient, algorithm.

Additional invariants

Another direction to explore is the definition of new invariants. In this work, we based the extraction of invariants one the cofibrant replacement, since cofibrant tame parametrised chain complexes are of finite representation type. Since all objects in $\mathbf{tame}\left([0,\infty)\,,\mathbf{ch}\right)$ are fibrant, the study the fibrant replacement does not provide more insightful information. But model category theory is much richer, leaving room for the study of other theoretical tools for retrieving invariants.

In conclusion, there are multiple directions that this work opened we believe are worth studying, both from the theoretical and the algorithmic point of view.

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