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Geometric and topological aspects of quantum defects

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Abstract

In this thesis we present a detailed study of the dynamics of closed, twisted quantum vortices in Bose-Einstein condensates already published by the author of the present work in two papers [54], [67]. The study of how geometric and topological features affect the dynamics of vortices is very important for the description of interactions between vortices and for their reconnections. We generalize twist for quantum defects defining the concept of *twist phase*. Then we find a modified Gross-Pitaevskii equation for the time evolution of a twisted vortex state. We discover that such a state is unstable and its evolution is dominated by a non-Hermitian Hamiltonian underlying the non-reversibility nature of the dynamics of twisted defects. Using the hydrodynamic description of Bose-Einstein condensates and applying Kleinert's theory to manage multi-valued phase fields, we find a complete set of integro-differential equations that quantitatively describe the dynamics of twisted vortices. Depending on the nature of the twist phase injected we propose two different stabilization mechanisms: if the twist phase is global then a secondary, central vortex is produced changing the linking number of the system. This mechanism can be seen as dominated by the presence of a topological phase and it is analyzed using Kleinert's theory. We thus prove theoretically what has been discovered numerically in [12]. In case of a local twist phase, no secondary vortex will form and the system produces unstable Kelvin waves with exponentially growing amplitude in regions where $\nabla^2\theta_{tw} > 0$. We demonstrate that to minimize the energy and to stabilize the system the vortex coils producing non-zero writhe and extinguishing its twist phase. This mechanism can be seen as produced by the effect of a geometric phase on the system. We also propose an experiment to inject a twist phase on a quantum vortex in order to prove or disprove such stabilization mechanisms.

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Contents

Chapter 1

Introduction

In recent years the concept of twisted vortices has obtained increasing attention within the physics community. On one hand, nowadays twist already plays an important role in many applications. For instance, optical vortices [1] can be created, manipulated and used in many fields, from imaging [2] to quantum information [3], [4], from new technologies to their use for the production of vortices in condensates [5], [6]. Optical vortices posses a twisted, helical phase profile and have interesting properties. Also in chemistry, with the observation of organizing centers made by scroll ring waves, twist has been found to have an important role [8]. In solid state physics twisted Skyrmions with important features have been observed [7]. On the other hand, the increasing attention given to quantum defects in Bose-Einstein condensates and their hydrodynamic characterization have permitted the application of geometric and topological tools, taken from classical vorticity theory, to study the dynamics and interactions between quantum defects both theoretically [10] and numerically [**?**]. Some of these concepts must be generalized to quantum fluids. One of these is the twist concept.

In this thesis we will concentrate on vortices in Bose-Einstein condensates. Twist in classical fluid dynamics and magnetohydrodynamics is a well known concept. However, whereas in classical fluid dynamics vortices have an internal structure made by many tangled vorticity lines that are suitable to define the twist concept, in Bose-Einstein condensates vortices are singularities of

phase that produce zero density lines in the condensate. These phase defects have no internal structure, therefore it has been difficult to generalize twist for them. In the paper [9] an interpretation is proposed for the generalization of twist that we will recall. The necessity of a theoretical and robust generalization of twist in quantum fluids and the study of its effects on the dynamics of quantum defects is necessary for a better comprehension of the evolution and of the interactions between them. Moreover in recent numerical simulations [12] it has been observed that a closed, twisted vortex in a condensate produces a secondary, central vortex linked with the first one. Thus, it seems that twist on a defect can change the linking number of the system and, consequently, its dynamics. In this thesis twist for quantum defects is theoretically generalized introducing the concept of *twist phase* and a novel theory is proposed that studies the dynamics of twisted quantum vortices in Bose-Einstein condensates and underlines its relation with geometric phase. A hypothetical experiment where twist phase is injected into a defect is also proposed

The first chapters are devoted to a review of known facts about Bose-Einstein condensates and geometric phase (apart from a result on writhe that we will demonstrate ex novo). In chapter five we will present a theory by Kleinert that will be used repeatedly throughout the thesis. Chapters seven and eight are the main results of the work and are devoted to a detailed analysis of the effects and dynamics of quantum vortices affected by a global or a local twist phase respectively, demonstrating also the numerical results found in [12].

Chapter 2

Bose-Einstein condensates

2.1 Cooling down a gas of bosons

When you cool down a diluted bosonic gas to very low temperature a new state of matter is realized in which all the bosons are in the same ground state. This gas of bosons is called a *Bose-Einstein condensate* (BEC) (see [13] and [14], [15]). At a temperature of $0\,K$ the degenerate ground state of the gas can be described by a unique field $\Psi = \Psi(\boldsymbol{x}, t)$, depending on position $\boldsymbol{x}$ of each particle and of time t, called the *order parameter*. It describes a BEC of weakly interacting bosons. The squared modulus of the order parameter describes the density of particles at a certain position and time: $|\psi(\boldsymbol{x}, t)|^2 = \rho(\boldsymbol{x}, t)$. From now on we will omit any spacial and time dependence unless when of particular importance.

Now we summarize the standard techniques to create and manipulate BECs. The main techniques used to cool down atoms and form a BEC are laser cooling and magnetic traps. Often both are used in the same experiment. Magnetic traps (see [16]) consist of a uniform magnetic field B_z aligned along a principal direction $\hat{z}$ and a perpendicular, spatially harmonic field $B_\perp$ due to a system of quadrupole magnets. The magnetic field has a minimum at the origin of the axes. To trap atoms (usually ^{23}Na or ^{87}Rb) in the minimum of the magnetic field, one has to minimize the particle's energy E. Since

$$\Delta E = -\boldsymbol{\mu} \cdot \boldsymbol{B}, \tag{2.1}$$

where $\boldsymbol{\mu}$ is the magnetic moment of each particle, atoms with low μ have minimum energy in higher magnetic field zones; they are called *high-field-seekers*. Atoms with high magnetic moments have minimum energy in low magnetic field zones. They are called *low-field-seekers*. Magnetic traps can trap only the low-field-seekers near the minimum and thus cooling down the system, by getting rid of all the other higher-energy particles. Magnetic trapping is more efficient than laser cooling because in the latter some momentum is transferred from photons to the atoms and then the cooling down process is limited. Since the magnetic moment is proportional to the spin of the atoms, one needs to make a transition between a low-spin atomic state to a high-spin, excited, atomic state. Usually one does this by a radio frequency transition.

Another fundamental technique, used in a typical experiment to manipulate BECs, are optical tweezers. These tools are composed of laser beams trapping neutral, dipole particles in their harmonic electric fields. Many different kinds of beams can be used: Gaussian beams, Gauss-Laguerre beams (transferring also orbital angular momentum, OAM) [6], etc. For Gaussian beams, atoms are trapped where the beam width is a minimum (waist, corresponding to a maximum of intensity of the beam and of the electric field).

We will see another method for cooling down atoms that uses the Berry phase in the chapter on geometric phase.

After a brief review of the principal methods for cooling down atoms and forming a BEC, in the next section we will analyze the dynamics of a Bose-Einstein condensate.

2.2 Gross-Pitaevskii equation

The equation that describes the dynamics of a BEC is the Gross-Pitaevskii equation (GPE):

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\Psi + \frac{1}{2}g\left(|\Psi|^2 - \mu\right)\Psi + V_{ext}\Psi \ , \tag{2.2}$$

where $\mathrm{i} = \sqrt{-1}$, $\hbar = h/2\pi$ (h Planck's constant), m_0 is the boson's mass, μ is the chemical potential. $V_{ext}(\mathbf{x}, t)$ an external potential, g the coupling constant, $g = 4\pi\hbar^2\frac{\lambda}{m_0}$, depending on the boson-boson scattering length λ, and it is repulsive for $g > 0$, attractive where $g < 0$. We will consider $g > 0$ that is the more interesting case.

$\Psi = \Psi(\mathbf{x}, t)$ is the complex wave function; $\mathbf{x}$ denotes the position vector of a point in the ambient space and t time. This is a non-linear Scrhödinger equation in three-dimensional space.

By standard re-scaling [17] we make the following substitutions:

$$\mathbf{x} \to \frac{\hbar}{\sqrt{2\, m_0\, V_{ext}}}\mathbf{x}' \tag{2.3}$$

$$t \to \frac{\hbar}{2V_{ext}}t'$$

$$\Psi \to \sqrt{\frac{V_{ext}}{g}}\,\psi.$$

Substituting these expressions into (2.2), redefining the derivative operators and dropping the primes, after some simple algebra the equation above reduces to the non-dimensional GPE in standard form, given by

$$\frac{\partial\psi}{\partial t} = \frac{\mathrm{i}}{2}\nabla^2\psi + \frac{\mathrm{i}}{2}\left(1 - |\psi|^2\right)\psi. \tag{2.4}$$

In what follows we will describe a BEC at zero temperature, fulfilling the entire space and such that $\rho \to 1$ as $|\boldsymbol{x}| \to +\infty$.

Furthermore we note that the GPE can be written as

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \hat{H}\psi, \tag{2.5}$$

where H is the Hamiltonian operator of the mean field ψ.

For other interesting applications and uses of BECs see, for instance, [17] and [18].

2.3 Total energy and momentum

The total energy of a BEC ([17]) can be computed from the non-dimensional GPE (2.4) computing the expectation value of the single-particle Hamiltonian evaluated in the order parameter ψ: $E = \langle\psi| H |\psi\rangle$. Therefore the total energy associated with a BEC is

$$E = \int \left(\frac{1}{2} |\boldsymbol{\nabla}\psi|^2 - \frac{1}{2}|\psi|^2 + \frac{1}{4}|\psi|^4 \right) d^3\boldsymbol{x}. \tag{2.6}$$

If ψ^* is the complex conjugate of the order parameter, the total linear momentum is ([17])

$$P = \frac{\mathrm{i}}{2} \int \left(\psi \boldsymbol{\nabla}\psi^* - \psi^* \boldsymbol{\nabla}\psi\right) d^3\mathbf{x}. \tag{2.7}$$

The total number of particles is

$$N = \int |\psi|^2 \, d^3\mathbf{x}. \tag{2.8}$$

It is possible to show that the GPE is invariant under Galilean transformations in an unbounded or periodic domain and hence the total energy, momentum and number of particle of a condensate are conserved ([17]). Moreover, it is simple to show that the GPE is invariant under a global $U(1)$ gauge transformation, even though it varies under local $U(1)$ gauge transformations.

2.4 Madelung transform

A useful representation of BECs is provided by *Madelung transform* of the order parameter (for a detailed review on the Madelung-Bohm derivation of hydrodynamics equations see [19]),

$$\psi(\boldsymbol{x}, t) = \sqrt{\rho}(\boldsymbol{x}, t) e^{i\chi(\boldsymbol{x},t)}, \tag{2.9}$$

where χ is the phase of the condensate. The Madelung transform has the advantage of describing the evolution of a condensate as an irrotational fluid with velocity $\mathbf{u} = \boldsymbol{\nabla}\chi$ and density ρ. Indeed, if we substitute the Madelung

transform into (2.4) and compute the derivatives we obtain

$$\left(\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t}+\mathrm{i}\,\sqrt{\rho}\frac{\partial\chi}{\partial t}\right)e^{i\chi}=\frac{\mathrm{i}}{2}\boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2\sqrt{\rho}}\boldsymbol{\nabla}\rho+\mathrm{i}\,\sqrt{\rho}\boldsymbol{\nabla}\chi\right)e^{i\chi}\right]+\frac{\mathrm{i}}{2}(1-\rho)\sqrt{\rho}\,e^{\mathrm{i}\,\chi} \tag{2.10}$$

$$=\frac{\mathrm{i}}{2}\left(-\frac{1}{4\rho\sqrt{\rho}}|\boldsymbol{\nabla}\rho|^2+\frac{1}{2\sqrt{\rho}}\nabla^2\rho+\frac{\mathrm{i}}{\sqrt{\rho}}\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\chi+\mathrm{i}\sqrt{\rho}\nabla^2\chi-\sqrt{\rho}|\boldsymbol{\nabla}\chi|^2\right)e^{i\chi}$$

$$+\frac{\mathrm{i}}{2}(1-\rho)\sqrt{\rho}\,e^{\mathrm{i}\,\chi}.$$

Simplifying and taking respectively the real and imaginary parts in both sides we obtain two equations for ρ and χ:

$$\partial_t\rho+\boldsymbol{\nabla}\cdot(\rho\,\mathbf{u})=0; \tag{2.11}$$

$$\partial_t\chi=-\frac{1}{2}|\mathbf{u}|^2-Q+\frac{1}{2}(1-\rho), \tag{2.12}$$

a continuity equation for the density ρ and a quantum version of the Hamilton-Jacobi equation for the phase χ, where $Q=-\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$ is the *quantum potential*. If we take the gradient of the second equation we obtain Navier-Stokes-like equations ([17]).

2.5 Vortices in BECs

BECs can have non-trivial ground states in which there are some points where the order parameter vanishes: $\psi=0$. Loci in which $\rho=0$, where phase is undefined, are lines in three-dimensional BECs. These phase defects are empty, singular lines γ. These null lines are called vortices because they are characterized by a quantized circulation along any path C encircling γ:

$$\Gamma(\gamma)=\oint_C\boldsymbol{\nabla}\chi\cdot\,d\boldsymbol{x}=2\pi\,n\qquad n\in\mathbb{Z}. \tag{2.13}$$

Thus, even though there is no vorticity, since the flow is not simply connected owing to $C=0$ it has an associated circulation around them. However, one

can treat a phase singularity as if there was a vorticity field $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ concentrated at the line. Since these vortices have a core of only some Ångström it is reasonable and useful to think of this vorticity field as a Dirac delta, singular field along γ, that is zero otherwise:

$$\boldsymbol{\omega}(\gamma, t) = \Gamma \boldsymbol{\delta}(\boldsymbol{x}, \gamma, t) = \Gamma \delta(\boldsymbol{x}, \gamma, t) \hat{T}, \tag{2.14}$$

where we introduced here the unit tangent vector to γ, $\hat{T}$. The notation used here is that of Kleinert [45] and will be explained in detail in chapter five.

Chapter 3

Solid angle

In his book [20] Maxwell rediscovers the relation from the scalar magnetic potential due to a magnetic shell and the solid angle subtended by the edge of the shell, already studied by Gauss and Thomson.

3.1 Relation between the scalar magnetic potential and the solid angle

There is a relation between the scalar magnetic potential V generated by a magnetic shell with edge γ and the solid angle Ω subtended by γ at a point $\boldsymbol{x}$ in space (Gauss 1839, [21]). Indeed, consider an infinitesimal surface element in a point $\boldsymbol{x}'$ on the magnetic shell, $d\boldsymbol{S}$. This infinitesimal surface element can be seen as composed by many magnetic dipoles with infinitesimal magnetic moment $d\boldsymbol{m}$. The strength of magnetization $\boldsymbol{\sigma}$ of the shell is the magnetization along the vector $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$ multiplied by the thickness of the shell. Since the magnetization measures how many dipoles there are per unit volume, we have that σ indicates how many dipoles there are in each surface element $d\boldsymbol{S}$. Since any infinitesimal magnetic moment can be associated and is proportional to the electric current I that generates the magnetization on the infinitesimal surface, $d\boldsymbol{m} = I\, d\boldsymbol{S}$, then we see that the strength of any surface element of the magnetic shell has dimensions of current, therefore we can write the strength towards direction $\hat{r}$ as $\boldsymbol{\sigma} = I\,\hat{r}$. Since magnetic dipoles

are formally equal to electric dipoles, then the infinitesimal scalar magnetic potential measured in a point in space $\boldsymbol{x}$ is

$$d\,V = \frac{\boldsymbol{\sigma} \cdot d\boldsymbol{S}}{r^2} = \frac{\sigma \cos\theta \,\mathrm{d\,S}}{r^2} = \frac{I \cos\theta \,\mathrm{d\,S}}{r^2}, \tag{3.1}$$

where θ is the angle between the outward normal to the infinitesimal surface and the separation vector $\boldsymbol{r}$ between the points $\boldsymbol{x}$ and $\boldsymbol{x}'$. But $\frac{dS}{r^2}\cos\theta = \mathrm{d}\Omega$, the infinitesimal solid angle subtended by $d\,S$ as seen from $\boldsymbol{x}$. If the magnetic strength of each elementary surface is constant then

$$V = \int_S \frac{\boldsymbol{\sigma} \cdot d\boldsymbol{S}'}{r^2} = I \int_S \frac{\hat{r} \cdot d\boldsymbol{S}'}{r^2} = I\,\Omega(\boldsymbol{x}, \gamma), \tag{3.2}$$

where $\boldsymbol{S}'$ indicates that the integration variable is $\boldsymbol{x}'$, any point on the magnetic shell. In [20] it is demonstrated that this law is valid also for any magnet that can be decomposed into a distribution of magnetic shells. In this case we have the same law as (3.2), but now the solid angle is that subtended by the whole magnet.

3.2 Maxwell's derivation of the solid angle formula

Now that we have highlighted the relation between scalar magnetic potential and the solid angle subtended by a magnetic shell, we review the concept of the solid angle as was obtained by Maxwell in [20], reviewed in [21] and in [22], with a starting point that is opposite to what was presented in the above section.

We can define the solid angle $\Omega(\boldsymbol{x})$ subtended by a magnetic shell bounded by γ as the scalar magnetic potential V felt by a magnetic particle in $\boldsymbol{x}$. This translates into the concept of work done in carrying the magnetic particle from a point at infinity to $\boldsymbol{x}$ following a particular path joining the two points. Since at infinity the solid angle subtended by γ is zero because far away from γ it seems shrunk to an infinitesimal closed line, at infinity the potential goes to zero.

Consider a frame at rest with the magnetic particle. In this frame the particle sees the edge of the shell γ changing. In particular, if we denote with s the arch-length parameter of γ and with u the arch-length parameter of the path followed by the particle, then the unit vector $\hat{T} = \frac{d\boldsymbol{x}'}{d\,s}$ seems to move infinitesimally in the direction opposite to the path of the particle. Then it spans an infinitesimal parallelogram of edges $\frac{d\boldsymbol{x}'}{d\,s}$ and $\frac{d\,\boldsymbol{x}'}{d\,u} = -\frac{d\,\boldsymbol{x}}{d\,u}$. The infinitesimal solid angle $d\,\Omega$ spanned by $\hat{T}$ as seen by $\boldsymbol{x}$ is given by the volume of the pyramid with vertex in $\boldsymbol{x}$ with respect to the observation sphere centered in $\boldsymbol{x}$ with radius $r = |\boldsymbol{x} - \boldsymbol{x}'|$. Moreover, $d\,\Omega$ must depend on the infinitesimal translation ds of points on γ and on the path followed by the particle, du. Hence, for the solid angle we can find the following relation

$$d\,\Omega = \Pi\, ds\, du. \tag{3.3}$$

We must find the quantity Π. The volume of the pyramid is given by $\frac{1}{3}r^3\, d\,\Omega$. Since the volume of a tetrahedron of edges $\boldsymbol{r}$, $\frac{d\,\boldsymbol{x}}{d\,u}$ and $\frac{d\,\boldsymbol{x}'}{d\,s}$ is given by the product

$$\boldsymbol{r} \cdot \left(\frac{d\,\boldsymbol{x}'}{d\,s} \times \frac{d\,\boldsymbol{x}}{d\,u} \right) ds\, du, \tag{3.4}$$

it is deduced that

$$\Pi = \frac{1}{r^3} \boldsymbol{r} \cdot \left(\frac{d\,\boldsymbol{x}'}{d\,s} \times \frac{d\,\boldsymbol{x}}{d\,u} \right) \tag{3.5}$$

and then

$$d\,\Omega(\boldsymbol{x}) = \frac{\boldsymbol{r} \cdot \left(\frac{d\,\boldsymbol{x}'}{d\,s} \times \frac{d\,\boldsymbol{x}}{d\,u} \right)}{r^3}\, ds\, du. \tag{3.6}$$

By integrating over the curve γ and the entire path followed by the magnetic particle we obtain

$$\Omega(\boldsymbol{x}, \gamma) = \int \frac{\hat{r}}{r^2} \cdot \left(\frac{d\,\boldsymbol{x}'}{d\,s} \times \frac{d\,\boldsymbol{x}}{d\,u} \right) ds\, du. \tag{3.7}$$

This integral does not depend on the surface of the shell but only on its boundary γ (Thomson, 1850, [21]), this is why we have written explicitly the dependence of the solid angle on γ.

If the initial point of the path of the magnetic particle is not at infinity but at another point $\boldsymbol{x}_0$, then we have

$$\Omega(\boldsymbol{x}, \gamma) = \int \frac{\hat{r}}{r^2} \cdot \left(\frac{d\,\boldsymbol{x}'}{d\,s} \times \frac{d\,\boldsymbol{x}}{d\,u} \right) ds\, du + \Omega_0, \tag{3.8}$$

where Ω_0 is the solid angle spanned by γ from the point of view of $\boldsymbol{x}_0$. In particular, if initially the particle is infinitesimally close to the magnetic shell in its negative portion (given by the orientation of γ with respect to the outward normal to the shell) and is carried on the positive part infinitesimally close to the shell, then the initial and final solid angles differ from 4π (Thomson, 1850, [21]). If we pass across the magnetic shell the solid angle jumps of a value of 4π. Therefore the solid angle is a multi-valued function, defined modulo 4π and if the particle path is such that it turns around the edge γ n times, then the solid angle differs by $4\pi n$ with respect to the solid angle relative to a path that does not pass through the surface of the shell. Therefore the solid angle can detect how many times a path turns around another path. If both the curves are closed this means that it can detect how much these curves are linked together (this was discovered, but left without proof and with no physical context, by Gauss in 1832 [21]. It had been rediscovered and proved by Maxwell in [20]).

3.3 Alternative formulation of the solid angle

In [22] a homotopy interpretation of the solid angle is given. For a closed curve γ in space, fix an observation point $\boldsymbol{x}$ from which to see the image of the curve on the observation unit sphere S^2 centered in $\boldsymbol{x}$. Describe the image of the curve on S^2 by the unit vector $\hat{r}(\boldsymbol{x})$. Fix any axis on S^2 and call it $\hat{r}_\infty$. The observation point and $\hat{r}_\infty$ does not depend on the curve. For simplicity let us take $\hat{r}_\infty$ coincident with the z-axis of S^2 (we can always perform such a rotation). Then, the solid angle subtended by the curve γ on the observation sphere as seen from $\boldsymbol{x}$ is given by:

$$\Omega(\boldsymbol{x}, \gamma) = \int \frac{\hat{r}_\infty \cdot (\hat{r} \times d\hat{r})}{r\ (1 + \hat{r}_\infty \cdot \hat{r})} \qquad \mod 4\pi. \tag{3.9}$$

Note that we have written the solid angle up to an integer multiple of 4π. This form of the solid angle can be demonstrated as done in [23] where it is demonstrated that

$$\int \frac{\hat{r}_\infty \cdot (\hat{r} \times d\hat{r})}{r\ (1 + \hat{r}_\infty \cdot \hat{r})} = \int \frac{\hat{r}}{r^2} \cdot d\,\boldsymbol{S}. \tag{3.10}$$

This equation says again that the solid angle depends only on the boundary γ and on the point of observation $\boldsymbol{x}$.

In the first part of the demonstration it is considered $(\hat{r}_\infty \cdot \hat{r}) \neq -1$ whereas in the second part the limiting case of equality is taken into account.

First part of the demonstration. It is sufficient to demonstrate that

$$\frac{\hat{r}}{r^2} = \boldsymbol{\nabla} \times \left[\frac{\hat{r}_\infty \times \hat{r}}{r\ (1 + \hat{r}_\infty \cdot \hat{r})} \right], \tag{3.11}$$

that is the integrand on the RHS of this equation plays the role of a vector potential $\boldsymbol{A}$ for the vector field $\frac{\hat{r}}{r^2}$. This is easily done by writing the vector potential in spherical coordinates (r, θ, ϕ) and remembering that, for simplicity, we have fixed $\hat{r}_\infty$ to be the z-axis:

$$\frac{\hat{r}_\infty \times \hat{r}}{r\ (1 + \hat{r}_\infty \cdot \hat{r})} = \frac{\sin\theta}{1 + \cos\theta} \hat{\phi}. \tag{3.12}$$

Now we write the curl operator in spherical coordinates and apply it to this vector potential. For the particular structure of the vector potential in spherical coordinates, the only term of the curl operator that does not vanish when deriving this is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\hat{r}}{r \sin\theta} \partial_\theta \left(A_\phi \sin\theta \right) = \frac{\hat{r}}{r^2}. \tag{3.13}$$

This is the demonstration when $\hat{r} \cdot \hat{r}_\infty \neq -1$, i.e. where the two unit vectors are not anti-parallel. Note that we are using the notation of [22] with a plus sign at the denominator of $\boldsymbol{A}$, instead of the notation of [23]. For this reason there is a minus sign of difference between our notation and that in [23] and this is why we require that the two unit vectors not to be anti-parallel.

The case of anti-parallel vectors, $\hat{r} \cdot \hat{r}_\infty = -1$, is mathematically treated in [23], with a suitable minus sign change. Here we give a physical demonstration. If the two vectors are parallel then we obtain the solid angle formula just demonstrated (3.9). If we turn from a parallel to an anti-parallel configuration it is as if we are carrying the point of observation $\boldsymbol{x}$ at an antipodal position with respect to the surface S. For what we have already said speaking of the magnetic shell, in such a situation the solid angle changes by 4π. Then the integral converges also in the case $\hat{r} \cdot \hat{r}_\infty = -1$, but to $\Omega(\boldsymbol{x}, \gamma) + 4\pi$.

If we project γ onto a observation unit sphere, i.e. normalizing the radius r of the sphere to unity, then we obtain the expression used in [22] for the solid angle:

$$\Omega(\boldsymbol{x},\gamma) = \int \frac{\hat{r}_\infty \cdot (\hat{r} \times d\hat{r})}{(1+\hat{r}_\infty \cdot \hat{r})} = \int \hat{r} \cdot d\boldsymbol{S} \qquad \mod 4\pi. \tag{3.14}$$

This expression for the solid angle has the well known physical interpretation of the flux of the magnetic field generated by a monopole in $\boldsymbol{x}$ i.e. an infinitesimal magnetic field line with one end in $\boldsymbol{x}$, and then a non-vanishing magnetic flux through any closed surface containing $\boldsymbol{x}$. If we close the surface S in such a way to contain the point $\boldsymbol{x}$ the flux of this monopole in a direction parallel to $\hat{r}_\infty$ is Ω, that flowing in the opposite direction is $\Omega + 4\pi$. Therefore we have different fluxes and thus two different vector potentials $\boldsymbol{A}$ circulating along γ depending on the direction with respect to $\hat{r}_\infty$. Hence the total flux through the closed surface is different from zero and 4π. The vector potential is multi-valued in $\boldsymbol{A}$. Indeed if we take the divergence of $\frac{\hat{r}}{r^2}$ and integrate over all the volume of the closed surface, by the divergence theorem we obtain the total flux and by Gauss' theorem we recover a delta function source $\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x})$ at $\boldsymbol{x}$. $\hat{r}_\infty$ plays the role of the Dirac's string (the equivalence of a cut-line) [24]. It is worth noting that the solid angle does not depend on $\hat{r}_\infty$. It is just an auxiliary axis to define it, as well as in the Dirac's string problem a line-cut needs to be fixed to recover a single valued vector potential.

We will see an analogous fact in Kleinert's theory but with multi-valued scalar potentials.

3.4 Linking number

We have seen that the solid angle can be seen as the magnetic potential felt by a magnetic particle at a position $\boldsymbol{x}$. This equals the work done in carrying the particle from infinity to $\boldsymbol{x}$. We have seen that the solid angle depends on the boundary of the magnetic shell. It depends on the path taken by the particle to arrive to its position only if this path goes across the magnetic

shell. Now if the particle travels along a closed path from starting and ending at $\boldsymbol{x}$, then the solid angle will change by 4π every time the path goes across the magnet and hence if it is linked with the boundary γ of the magnetic shell. Therefore, after a complete turn of the particle, the difference between the final solid angle accumulated along the path and the initial one differs only by $4\pi\, n$ where $n \in \mathbb{Z}$ is the number of times it intersects the area spanned by γ. It quantifies the degree of linking between the two curves, γ and the path. Therefore, given two curves γ_1 and γ_2 one can define their linking number as the integer number that quantifies the degree of linkage of the two curves. Hence we have the following

Definition 3.4.1 (Linking number)**.** If we describe the two curves by the position vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively then, given the vector $\boldsymbol{r} := \boldsymbol{x}_1 - \boldsymbol{x}_2$, the linking number between the two curves $Lk(\gamma_1, \gamma_2)$ is defined as

$$Lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \frac{\boldsymbol{r}}{r^3} \cdot (d\,\boldsymbol{x}_1 \times d\,\boldsymbol{x}_2). \tag{3.15}$$

The linking number is a topological invariant of the two vortices. Instead, when we have a single thin vortex we can measure the degree of knottedness by its *self-linking number* as we will see.

Chapter 4

Geometric phase

4.1 Berry Phase

In this chapter we will define the concept of geometric phase that will be very important in the analysis of the dynamics of twisted quantum vortices. For some fundamentals of quantum mechanics the reader can refer to the appendix.

Consider a Hamiltonian $\hat{H}$. Let it depend upon a set of parameters $\boldsymbol{X}$, which can represent a vector in a manifold of parameters M. For each fixed value of the vector parameter $\boldsymbol{X}$ consider a complete set of discrete energy eigenstates $|e_n(\boldsymbol{X})\rangle$ that are solutions of the following eigenvalue problem (see [27]. For a general point of view on geometric phase see [28] or [29]):

$$\hat{H}\,|e_n(\boldsymbol{X})\rangle = E_n(\boldsymbol{X})\,|e_n(\boldsymbol{X})\rangle\,, \tag{4.1}$$

where E_n are the eigenenergies associated with any eigenstate of the Hamiltonian. Let us consider only non-degenerate eigenstates (i.e. different eigenstates have different energies) and let the function $\boldsymbol{X} \to |e_n(\boldsymbol{X})\rangle$ be single-valued for fixed n, i.e. there exists (at least locally) a function $f : M \to H_n(\boldsymbol{X})$, where $H_n(\boldsymbol{X})$ is the n-th energy eigenspace. Now vary the parameter vector slowly in time: $\boldsymbol{X} = \boldsymbol{X}(t)$, so that any eigenstate and eigenenergy depends on time t with the parameter vector: $|e_n(\boldsymbol{X}(t))\rangle = |e_n(t)\rangle$. Geometrically it means to consider a curve $\gamma_t : t \to \boldsymbol{X}(t)$ on the parameters manifold M. Imagine to change the parameters such that at a final time

T $\boldsymbol{X}(T) = \boldsymbol{X}(0)$, i.e. parameters form a closed curve in M. If the change is very slow, then one can use the adiabatic theorem that ensures that the state $|e_n(t)\rangle$ remains an eigenstate of the Hamiltonian with energy $E_n(t)$. After a complete cycle the initial eigentsate is in the same eigenspace $H_n(\boldsymbol{X})$ but can differ from the initial one by an unobservable phase factor $e^{i\phi_n(\boldsymbol{X})}$. This means that while γ_t is closed in M, $f(\gamma_t)$ is an open curve in $\mathcal{H}$ (see appendix A for notations). For time varying Hamiltonian operators the evolution operator is given by (A.10). Since the evolution operator depends on the Hamiltonian operator, they can be simultaneously diagonalized. Hence, a generic eigenstate $|e_n\rangle$ in the n-th eigenspace, after a period T will have acquired a phase that depends on its evolution in time given by the Hamiltonian $\hat{H}(t)$:

$$|e_n(T)\rangle = e^{i\phi_{dyn}} |e_n(0)\rangle , \qquad \phi = -\frac{1}{\hbar}\int_0^T E_n(t)dt. \tag{4.2}$$

ϕ_{dyn} is called *dynamical phase* because it is acquired by a state every time it evolves in time, its evolution being given by the Hamiltonian.

However, Berry ([27]) demonstrated that there is an additional change in phase due to the geometry of the closed path γ in M. In general, a state $|\psi(t)\rangle$ can be written as a linear combination of eigenstates of energy $|e_n(t)\rangle$. If we assume that initially the state coincides with the $n-$th energy eigenstate (it is always possible to prepare such a state) $|\psi(0)\rangle = |e_n(0)\rangle$, the evolution of the system due to a time-varying energy can make the eigenstate $|e_n(0)\rangle$ jump to a different one at time t: $|e_m(0)\rangle \quad m \neq n$. However, as we have already underlined, if the evolution is sufficiently slow we can apply the *adiabatic approximation*.

Remark. Under adiabatic approximation, the state $|\psi(t)\rangle$ remains proportional to the $n-$th energy eigenstate at time t:

$$|\psi(t)\rangle = c_n(t) |e_n(t)\rangle . \tag{4.3}$$

Let us use the adiabatic approximation and substitute the state at time t, $|\psi(t)\rangle$, in the Scrhödinger's equation:

$$\frac{d}{dt}c_n(t)\ |e_n(t)\rangle + c_n(t)\frac{d}{dt}|e_n(t)\rangle = \frac{-i}{\hbar}E_n(t)c_n(t)\,|e_n(t)\rangle . \tag{4.4}$$

The inner product with another energy eigenstate $|e_m(t)\rangle\,, \quad m \neq n$ and the normalization of the eigenstates leads to the following necessary and sufficient condition for adiabaticity:

$$\langle e_m(t)|\frac{d}{dt}e_n(t)\rangle = 0, \tag{4.5}$$

that means that, under the adiabatic approximation, the change of an eigenstate with respect to another one must be negligible. By computing the inner product with the same state $|e_n(t)\rangle$ into (4.4) one obtains a pure phase factor:

$$c_n(t) = e^{i\phi_{dyn}} e^{i\int_0^t \langle e_n(t')|\frac{d}{dt'}|e_n(t')\rangle d\,t'}, \tag{4.6}$$

i.e.

$$|\psi(t)\rangle = e^{i\phi_{dyn}} e^{i\int_0^t \langle e_n(t')|\frac{d}{dt'}|e_n(t')\rangle d\,t'}\,|e_n(t)\rangle\,. \tag{4.7}$$

The term

$$\int_0^t \langle e_n(t')|\,\frac{d}{dt'}\,|e_n(t')\rangle\,d\,t' = \int_{\boldsymbol{X}(0)}^{\boldsymbol{X}(t)} \langle e_n(\boldsymbol{X}(t'))|\,\frac{d}{d\boldsymbol{X}(t')}\,|e_n(\boldsymbol{X}(t'))\rangle\,d\,\boldsymbol{X}(t') = \tag{4.8}$$

$$\int_{\boldsymbol{X}_0}^{\boldsymbol{X}} \langle e_n(\boldsymbol{X}')|\,\frac{d}{d\boldsymbol{X}'}\,|e_n(\boldsymbol{X}')\rangle\,d\,\boldsymbol{X}' \tag{4.9}$$

is independent of time and on the Hamiltonian of the system. If the path in parameter space were not closed, for the gauge freedom one could choose a gauge such that $\langle e_n(t')|\,\frac{d}{dt'}\,|e_n(t')\rangle = 0$ leading to unity for the extra phase contribution. However we are considering cyclic evolutions, hence the path γ in M is closed. We define

$$\phi_{geo} := \int_\gamma \langle e_n(\boldsymbol{X}')|\,\frac{d}{d\boldsymbol{X}'}\,|e_n(\boldsymbol{X}'\rangle\,d\,\boldsymbol{X}' \tag{4.10}$$

and call it *Berry phase*. It depends only on the geometry of the curve in M and thus is also called *geometric phase*. Berry phase has a huge amount of applications in a very wide range of physical phenomena. To see only some of the many applications in physics see [30], [31], [32], [33], [34], [35]; for the classical mechanics counterpart, the *Hannay's angle*, see [36]; for applications

in condensed matter physics see for instance [37]; for applications in solid state physics see [38]. Berry demonstrated that it is gauge invariant and, therefore, an irremovable and measurable quantity. Hence the total phase shift after a travel along a closed path in parameter space is the sum of two terms: a dynamical phase due to the time evolution and a geometric phase which depends only on the path in parameter space and it is gauge invariant only when the path is closed:

$$|\psi(T)\rangle = e^{i\phi}\,|\psi(0)\rangle\,, \qquad \phi = \phi_{dyn}+\phi_{geo} = \frac{1}{\hbar}\int_0^T E_{n(t)}dt + \int_\gamma \langle e_n|\,\boldsymbol{\nabla}\,|e_n\rangle\cdot d\boldsymbol{X}. \tag{4.11}$$

4.2 Aharonov-Anandan phase

After Berry, Aharonov and Anandan ([39]) showed that a geometric phase exists for cyclic evolutions, also for non-adiabatic processes.
Consider a state $|\psi(t)\rangle \in \mathcal{H}$ normalized, i.e. $\langle\psi|\psi\rangle = 1$. Schrödinger's equation gives the time evolution of $|\psi(t)\rangle$. It defines a path $t \to \psi(t)$ on $\mathcal{H}$. This path projects to a path in the physical Hilbert space H_{phys}. In this space any state can be thought as a projector P, an operator that projects any state into the fixed state (modulo a pure phase):

$$|\psi\rangle \to \hat{P} := |\psi\rangle\,\langle\psi|\,. \tag{4.12}$$

The time evolution for P, is given by Heisenberg's equation:

$$\frac{i}{\hbar}\frac{d}{dt}\hat{P} = [\hat{H}, \hat{P}]. \tag{4.13}$$

These kind of projectors are also called density operators and used in entanglement theory with mixed states. If the evolution is cyclic then, at a time T, $\hat{P}(T) = \hat{P}(0)$. In this case the path formed is closed in $\mathcal{H}_{phys}$ and we call it γ. This does not prevent, anyway, the path to be closed also in the starting Hilbert phase $\mathcal{H}$. In general the gauge freedom on the choice of a non-measurable phase implies that

$$|\psi(T)\rangle = e^{i\phi}\,|\psi(0)\rangle\,. \tag{4.14}$$

The evolution in time is unperturbed if we modify the Hamiltonian adding an operator proportional to the identity and that changes in time. This is the only possibility to preserve the linearity of the Hamiltonian. We can thus define the following new Hamiltonian operator (no condition on adiabaticity is asked)

$$\hat{H} + id\, a(t), \tag{4.15}$$

where $a(t)$ is any real function of time. It is immediate to note that this does not influence the evolution of $\hat{P}(t)$ the physical Hilbert space because every operator commutes with the identity operator. The state changes by a phase $\frac{1}{\hbar}\int_0^T a(t)dt$ due to evolution of the Hamiltonian in time. Hence:

$$|\psi'(T)\rangle = e^{-i\frac{1}{\hbar}\int_0^T a(t)dt}\,|\psi(t)\rangle\,. \tag{4.16}$$

Hence

$$|\psi'(T)\rangle = e^{i\phi'}\,|\psi(0)\rangle\,. \tag{4.17}$$

This leads

$$\phi' = \phi - \frac{1}{\hbar}\int_0^T a(t)dt. \tag{4.18}$$

Since ϕ' is free, one can choose it to be 0 and hence this lead to $\phi = \frac{1}{\hbar}\int_0^T a(t)dt$. Let us multiply both sides of the Schrödinger equation by $|\psi'(t)\rangle$ and integrate in time. We obtain

$$\int_0^T \langle\psi'(t)|\,\frac{d}{dt}\,|\psi'(t)\rangle\,dt = \frac{1}{\hbar}\int_0^T \langle\psi'(t)|\,\hat{H}\,|\psi'(t)\rangle\,dt + \frac{1}{\hbar}\int_0^T a(t)dt. \tag{4.19}$$

We see that the total phase ϕ acquired by the wave function travelling along γ is decomposed into a term depending on the Hamiltonian and one very similar to that of (4.10). The first phase is again called dynamical phase ϕ_{dyn}. The LHS phase does not depend on time but only on the geometry of γ and it is called *Aharonov-Anandan phase* ϕ_{geo} and it is gauge invariant under phase choice, hence it cannot be canceled by a different choice of ϕ'. No assumption has been made on $\hat{H}$ and on adiabaticity, hence we can conclude that in general cyclic evolutions a system acquires a measurable phase that depends on the geometry of the path (that can be determined by time evolution or any other parameter). This is a more general phase with

respect to the Berry phase. Both these phases are often referred to with the general name of *geometric phase*. Therefore, finally we have the following decomposition even for cyclic, non-adiabatic processes:

$$\phi = \phi_{dyn} + \phi_{geo}. \tag{4.20}$$

4.3 Example: spin one-half particle under a slowly varying magnetic field

Let us consider a simple example: a quasi-classical particle with spin $\frac{1}{2}$ precessing around a magnetic field line slowly varying in time [40]. The Hamiltonian operator is

$$\hat{H} = \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{\sigma}, \tag{4.21}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{B}$ are the Pauli matrices and the magnetic field respectively. One can always choose the magnetic field along the z-axis and $\sigma = \sigma_z$. The eigenstates are those of the spin one-half: $|+\rangle$ for the spin up and $|-\rangle$ for the spin down. These two states are normalized:

$$\begin{aligned} \langle +|+\rangle = \langle -|-\rangle = 1; \\ \langle +|-\rangle = \langle -|+\rangle = 0 \end{aligned} \tag{4.22}$$

A general normalized state can be written as: $|\psi\rangle = cos(\frac{\theta}{2})\,|+\rangle + sin(\frac{\theta}{2})\,|-\rangle$. Substituting these states into the Schrödinger equation one can prove that a general solution is $|\psi(t)\rangle = e^{-i\frac{Bt}{2\hbar}} cos(\frac{\theta}{2})\,|+\rangle + e^{i\frac{Bt}{2\hbar}} sin(\frac{\theta}{2})\,|-\rangle$. The state differs only by a pure phase with the initial state after a time $T = \frac{2\pi\hbar}{B}$. In this case $|\psi(T)\rangle = e^{-i\pi}\,|\psi(0)\rangle$. Hence, the total phase acquired after a turn is $-\pi$. By calculating the expectation value of the Hamiltonian operator and integrating as in (4.19), it is possible to prove that $\phi_{dyn} = -\pi cos\theta$. Therefore $\phi_{geo} = -\pi - \phi_{dyn} = -\pi + \pi cos\theta = \pi(cos\theta - 1) \quad (mod 2\pi)$. This is exactly minus half the solid angle on the unit sphere of the closed curve created by the precessing spin under the effect of the slowly varying magnetic field.

The relation between geometric phase and solid angle is general. One can demonstrate that if the spin is m and $\Omega(\gamma)$ is the solid angle subtended by γ on the unit sphere, then $\phi_{geo} = -m\Omega$ (see [29] for more details and for the demonstration for the general case of a particle of spin k).

4.4 Applications in BECs: creation of quantum vortices using Berry phase.

One method to create vortices in BECs is to induce a Berry phase to the atoms of the condensate by inverting the $z-$component of the magnetic field of a Ioffe-Pritchard magnetic trap (see [41]) and so rotating their total angular momenta. As explained in the paper, the trap used consists of a magnetic field $\boldsymbol{B} = B_z\,\hat{z} + B'\,(x\,\hat{x} + y\hat{y})$, where B_z and B', the radial gradient of B, are constant. The perpendicular component is a quadrupole field. Each atom (^{23}Na) is prepared in a $|F = 1, m_z = -1\rangle$ state, where F is the total atomic angular momentum and m_z represents its z-projection. The atoms are then held in optical tweezers and conducted in a chamber with an Ioffe-Pritchard magnetic trap, and part of them is excited to the $|2, +2\rangle$ state by a radio frequency transition $|1, -1\rangle \rightarrow |1, 0\rangle \rightarrow |1, +1\rangle$ (100% probability transition) and a subsequent microwave transition $|1, +1\rangle \rightarrow |2, +2\rangle$ (80% probability transition). The experiment is done with both the $|1, -1\rangle$ and $|2, +2\rangle$ condensates. The field B_z is inverted adiabatically, changing the z-component of atoms' angular momenta but thus mantaining the atoms in the same eigenstates of thir initial total angular momenta states. The latter undergo a rotation of π around their rotation axes in a frame co-moving with each atom. The author starts from a $B_z \gg B'\,r > 0$ and end with $B_z \ll B'\,r < 0$ ($r = x^2 + y^2$) so that, at the beginning $\boldsymbol{F}$ is parallel to $\hat{z}$ and in the end is antiparallel.

All rotation axes $\hat{n}$ of $\boldsymbol{F}$ are orthogonal to $\hat{z}$ and to $\boldsymbol{B}_\perp$, the component of the magnetic field in the radial and azimuthal directions. In particular, the

directions of $\hat{n}$ depend on the azimuthal angle ϕ:

$$\hat{n}(\phi) = sin\,(\phi)\,\hat{x} + cos\,(\phi)\,\hat{y}. \tag{4.23}$$

This means that there is a precession of $\boldsymbol{F}$ with respect to ϕ when the field is inverted. Thus, every angular momentum of each atom is rotated by a π angle in a frame co-moving with each atom, but in the laboratory frame, each angular momentum will rotate differently depending on the ϕ position of each atom. Since the angular momentum generates rotations of the atoms (inner or outer), then an atom will have a phase shift with respect to another one due to their relative ϕ positions on the plane perpendicular to $\hat{z}$ (see pictures in the cited paper). This phase shift depends only on the geometric path C done by $\boldsymbol{F}$ in the parameter space, and thus it is a Berry phase $\gamma(C)$. The authors represents the rotations of the angular momenta of two atoms on the unit sphere putting $\boldsymbol{F}$ at the common origin of the sphere. When the field is inverted, two different $\boldsymbol{F}$ will draw two different paths, both from the north pole to the south pole. The phase shift is proportional to the solid angle of the closed path formed by the two different lines:

$$\gamma(C) = -m_F\Omega(C) = -2\,m_F(\phi_1 - \phi_2), \tag{4.24}$$

where m_F indicates the (in absolute value) maximum value of m_z and ϕ_i indicates the ϕ position of the i-th atom. Thus:

$$\gamma(\phi) = -2\,m_F\phi. \tag{4.25}$$

Therefore, the misalignment of the rotation axes causes a precession of the angular momenta with a consequently local phase shift of the condensate, given by a rotation of the condensate: $\psi \to e^{-2\,m_F\phi}\psi$. This is evidently a vortex state at the origin, with circulation given by $2\,m_F$. It is evident that this geometric phase is, in fact, a topological one, because it depends only on the domain of ϕ. A vortex forms in the region where ϕ is not defined.
The authors detect the vortex by an axial absorption imaging, with a resonant laser light, and then imaged by a CCD camera.

4.5 Topological phase: Aharonov-Bohm effect

Aharonov-Bohm effect (AB effect) [42], [43] is a phenomenon in which the vector potential $\mathbf{A}$, produced by a thin, current-carrying solenoid in the space surrounding it, acquires physical meanings producing measurable effects on charge particles traveling in the surroundings of the solenoid, even though outside the solenoid region the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \mathbf{A} = 0$. The vector potential $\mathbf{A}$ is not defined at the center of the solenoid. The precence of this singularity makes the domain $\mathcal{D}(\mathbf{A})$ of $\mathbf{A}$ multiply-connected. $\mathcal{D}(\mathbf{A})$ can then be divided in simply-connected components in each one of which $\mathbf{A} = \boldsymbol{\nabla}\phi$, ϕ being a scalar potential which is multi-valued because it is defined locally up to a constant function. Indeed $\mathbf{A}$ does not change if we make the change $\phi \to \phi + f, \quad f = const.$. If two particles travel toward a screen behind the solenoid following two different paths, on the left and on the right of the solenoid respectively, when they will interfere at the screen the usual fringe pattern will be changed because the two particles will have acquired a different phase passing at different sides of the solenoid. This is seen by considering a closed path γ encircling the solenoid. The phase shift acquired by the particle is equal to that acquired by a unique particle traveling along this closed loop and feeling the $\mathbf{A}$ potential. The phase shift is $\Delta\,\phi$ and is proportional to the magnetic flux passing through the area S enclosed by γ. Indeed the phase shift is proportional to the non-zero circulation of $\mathbf{A}$ around the solenoid $\Gamma(\gamma)(\mathbf{A})$ and for Stokes' theorem:

$$\Gamma(\gamma)(\mathbf{A}) = \int_\gamma \mathbf{A} \cdot d\,\boldsymbol{x} = \int_\gamma \boldsymbol{\nabla}\phi \cdot d\,\boldsymbol{x} = \Delta\phi = \int_S \boldsymbol{B} \cdot d\,\boldsymbol{S} \neq 0. \tag{4.26}$$

We therefore note that this phase shift is possible only because there is a singularity of the domain of the vector field $\mathbf{A}$, given by a singular magnetic field and, as a consequence, there is a multi-valued phase field ϕ that jumps when the path closes. The phase shift depends only on the jump, and then only on initial and final positions of the path and on how many times it travels around the solenoid, it does not depend on the geometry of the path.

Therefore in the case of the AB effect the extra phase acquired by the system is of topological nature. It is called a *topological phase*.
For applications of AB effect in random waves see [44].

Chapter 5

Kleinert's theory

In what follows we will need to manage multi-valued phases of a Bose-Einstein condensate, therefore in this chapter we will describe the theory explored by Kleinert and exposed in [45], which is very useful to treat multi-valued field in physics without the unnecessary complications of multi-valued function analysis or any theory of inclusions. However we note that Kleinert's theory is in line with other works that treats phase fields in presence of vortex singularities. This theory is generally known as *ϕ-mapping theory* [46], [47]. Further important, mathematical contributions in this direction are in [48]. However, from our modest viewpoint, we believe that Kleinert's theory is the more immediate and appropriate for our goals.

Phases and angles are famous examples of multi-valued, scalar fields. In particular, in classical electrodynamics the solid angle subtended by a current-carrying wire or that spanned by a magnetic field is a multi-valued function. In a Bose-Einstein condensate with a closed phase defect line γ, any surface of constant phase (isophase) is bounded by γ. Hence, the phase of the condensate depends on γ in the same way the solid angle depends on the geometry of the wire.

Now, consider a BEC having phase χ, with a phase defect line γ in it. The gradient $\boldsymbol{\nabla}\chi$ is the vector normal to any isophase surface. This is a velocity field. In [45] it is demonstrated that the gradient of a multi-valued function produces a singular vorticity field, despite the fact that the gradient field is

curl-free. This fact was clear to Maxwell [20]: the vector potential $\boldsymbol{A}$ produced at a point by a singular, closed magnetic field line can be written as the gradient of the multi-valued solid angle Ω subtended by the field line:

$$\boldsymbol{A} = \phi \boldsymbol{\nabla} \Omega, \tag{5.1}$$

where ϕ is the magnetic flux through any normal surface.

Since the phase χ of the condensate at a point $\boldsymbol{x}$ is multi-valued, and for what we have said above, it plays the role of the solid angle in a condensate with a singular vorticity line. Since we have that the velocity produced by the vortex is (fixing time) $\mathbf{u} = \frac{\hbar}{m} \boldsymbol{\nabla} \chi(\boldsymbol{x})$, where m is the mass of each particle of the condensate, it is the gradient of a multi-valued phase, because the values of χ jump by an integer multiple of 2π at any turn around the singularity. To easily manage multi-valued phases we must fix a cut surface S, which is itself an isophase surface hinged to the vortex line. From now on we will set all the dimensional constants equal to unity. If we fix a surface S bounded by the vortex, by Biot-Savart law it is possible to demonstrate (see [45], page 113 and foll., where the demonstration is given using solid angle) that

$$\chi(\boldsymbol{x}, S) = \int_S \frac{\boldsymbol{R'} \cdot d\, \boldsymbol{S'}}{R'^3}, \tag{5.2}$$

where $\boldsymbol{R'} := \boldsymbol{x'} - \boldsymbol{x}$ is the position vector between the point $\boldsymbol{x}$ and the surface S. Then, by Maxwell's theory,

$$\mathbf{u}(\boldsymbol{x}, S) = \boldsymbol{\nabla} \chi(\boldsymbol{x}, S). \tag{5.3}$$

Now, we have

$$\mathbf{u}(\boldsymbol{x}, S) = \boldsymbol{\nabla} \chi(\boldsymbol{x}, S) = \int_S \boldsymbol{\nabla} \left(\frac{\boldsymbol{R'} \cdot d\, \boldsymbol{S'}}{R'^3} \right) = - \int_S \boldsymbol{\nabla'} \left(\frac{\boldsymbol{R'} \cdot d\, \boldsymbol{S'}}{R'^3} \right), \tag{5.4}$$

where $\boldsymbol{\nabla'}$ stands for the gradient with respect to the variable $\boldsymbol{x'}$. Written in components it becomes

$$\mathbf{u}(\boldsymbol{x}, S) = \boldsymbol{\nabla} \chi(\boldsymbol{x}, S) = - \int_S \partial'_i \left(\frac{R'_j}{R'^3} \right) d\, S'_j, \tag{5.5}$$

where summation is understood for repeated indices. Summing and subtracting the term $\partial'_j \left(\frac{R'_i}{R'^3}\right) d\,S'_i$ we obtain

$$-\int_S \partial'_i \left(\frac{R'_j}{R'^3}\right) d\,S'_j = \tag{5.6}$$

$$-\left[\int_S \left(\partial'_i \left(\frac{R'_j}{R'^3}\right) d\,S'_j - \partial'_j \left(\frac{R'_j}{R'^3}\right) d\,S'_i\right) + \int_S \partial'_j \left(\frac{R'_j}{R'^3}\right) d\,S'_i\right] =$$

$$-\left[\int_S \left(d\,S'_j \partial'_i - d\,S'_i \partial'_j\right) \left(\frac{R'_j}{R'^3}\right) + \int_S \partial'_j \left(\frac{R'_j}{R'^3}\right) d\,S'_i\right]$$

The second integral is that of the divergence of a inverse-quadratic field, like the electric field generated by a point charge placed on the infinitesimal area $d\,S_i$. For Gauss theorem this is equal to the density charge. Thus, in this case of point charge, we have

$$\partial'_j \left(\frac{R'_j}{R'^3}\right) = 4\pi\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'). \tag{5.7}$$

Hence, we can see the second integral as the contribution of infinite many point-like sources placed on the surface S. This gives a source field that is different from zero only on S. Hence we define

$$\boldsymbol{\delta}(\boldsymbol{x}, S) = \int_S \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, d\boldsymbol{S}. \tag{5.8}$$

Now we give a complete demonstration of the decomposition formula given by Kleinert for the velocity only outlined by him. In coordinates, Stokes' theorem

$$\int_S \left[\boldsymbol{\nabla}' \times \left(\frac{\boldsymbol{R}'}{R'^3}\right)\right] \cdot d\,\boldsymbol{S}' = \int_{\gamma=\partial\, S} \frac{\boldsymbol{R}'}{R'^3} \cdot d\boldsymbol{x}' \tag{5.9}$$

becomes

$$\int_S \epsilon_{lmn} d\,S'_l \partial'_m \left(\frac{R'_n}{R'^3}\right) = \int_{\gamma=\partial\, S} \frac{R'_n}{R'^3}\, d\,x'_n. \tag{5.10}$$

This is true independently of the choice of the vector $\boldsymbol{R}'$. Thus the following equality holds:

$$\int_S \epsilon_{lmn} d\,S'_l \partial'_m = \int_{\gamma=\partial\, S} d\,x'_n. \tag{5.11}$$

Multiplying both sides of the equation by ϵ_{nij} and taking into account the fact that $\epsilon_{lmn} = \epsilon_{nlm}$, the equation becomes

$$\int_S \epsilon_{nij}\epsilon_{nlm} d\, S'_l \partial'_m = \int_{\gamma=\partial\, S} \epsilon_{nij} d\, x'_n. \tag{5.12}$$

Taking account of the ϵ symbol relation with the Kroneker delta $\epsilon_{nij}\epsilon_{nlm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, we have

$$\int_S (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, d\, S'_l \partial'_m = \int_{\gamma=\partial\, S} \epsilon_{nij} d\, x'_n. \tag{5.13}$$

Finally we arrive at the following relation:

$$\int_S \left(d\, S'_i \partial'_j - d\, S'_j \partial'_i\right) = \int_{\gamma=\partial\, S} \epsilon_{ijk} d\, x'_k, \tag{5.14}$$

where we have changed the name of the summed index n with k. Therefore, substituting what we have found in (5.6) we have

$$\mathbf{u}(\boldsymbol{x}, S) = \int_\gamma \frac{d\boldsymbol{x}' \times \boldsymbol{R}}{R^3} - 4\pi\boldsymbol{\delta}(\boldsymbol{x}, S). \tag{5.15}$$

The first term in the rhs of the equation is the Biot-Savart formula for the velocity $\mathbf{u}(\boldsymbol{x})$ produced by the vorticity line γ, that depends only on the point $\boldsymbol{x}$. Therefore we obtain the final relation (see [45] eq.4.31 pag.115 with Ω instead of χ.)

$$\mathbf{u}(\boldsymbol{x}) = \boldsymbol{\nabla}\chi(\boldsymbol{x}, S) + \boldsymbol{\delta}(\boldsymbol{x}, S). \tag{5.16}$$

Note that the velocity does not depend on the choice of the particular discontinuity surface S, but only on the presence of the vortex line γ and we can always re-define the phase $\chi \to \frac{\chi}{4\pi}$ such that 4π in front of $\boldsymbol{\delta}(\boldsymbol{x}, S)$ is absorbed to give the right constant for the circulation. Thus, if we want to compute $\mathbf{u}$ correctly, we have to take care of the contribution of the cut surface to the velocity, that Kleinert demonstrates to be proportional to a delta function on S. The value of $\mathbf{u}$, written in this way, is still that due to Biot-Savart, but re-arranged to take care of the multi-valuedness of the phase.

Now the velocity, as the gradient of a multi-valued phase, is decomposed into the sum of a gradient of a single-valued, discontinuous function dependent of

the cut surface, and of a Dirac delta function defined only on the cut surface. The adding of a delta function on the cut surface is necessary in order to make the expression for the velocity independent of the choice of the cut S. Kleinert ([45], pag. 115) demonstrates that, being

$$\boldsymbol{\delta}(\boldsymbol{x},\gamma) := \int_{\gamma} \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\, d\boldsymbol{x}', \tag{5.17}$$

then

$$\boldsymbol{\delta}(\boldsymbol{x},\gamma) = \boldsymbol{\nabla} \times \boldsymbol{\delta}(\boldsymbol{x},S), \tag{5.18}$$

where now the vortex γ is seen as the boundary of S. Indeed, if we rewrite in coordinate

$$\boldsymbol{\delta}(\boldsymbol{x},S) = \int_{S} \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')d\, S_l', \tag{5.19}$$

then, writing the curl using the Levi-Civita ϵ symbol and using (5.11) we obtain

$$\boldsymbol{\nabla} \times \boldsymbol{\delta}(\boldsymbol{x},S) = \int_{S} \epsilon_{lmn} d\, S_l'\, \partial_m \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = \int_{\gamma} \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') d\boldsymbol{x} = \boldsymbol{\delta}(\boldsymbol{x},\gamma). \tag{5.20}$$

This delta vector field $\boldsymbol{\delta}(\boldsymbol{x},\gamma)$ points along the tangent vector $\hat{T}$ of γ. Therefore, the gradient of a multi-valued function is not curl-free and its curl is exactly the phase defect line:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\theta(\boldsymbol{x}) = \boldsymbol{\delta}(\boldsymbol{x},\gamma). \tag{5.21}$$

This means that any isophase surface meets in γ, where the phase is undefined, and rotates around it generating a circulation. We now demonstrate that the velocity $\mathbf{u}(\boldsymbol{x})$ is, as expected, invariant under change of cut-surface S and that, beside being invariant under added gradients of single-valued functions to the velocity field, equation (5.21) is also independent on the choice of the cut surface S. Indeed, Kleinert states that if we changed the cut surface $S \to \tilde{S}$ the delta function (5.8) would change by a gradient of the delta of the volume V, $\delta\boldsymbol{x}, V := \int_V \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\, d\boldsymbol{x'}^3$ swept by S when changes to become $\tilde{S}$:

$$\boldsymbol{\delta}(\boldsymbol{x},\tilde{S}) = \boldsymbol{\delta}(\boldsymbol{x},S) + \boldsymbol{\nabla}\delta(\boldsymbol{x},V). \tag{5.22}$$

Now we give another demonstration of what stated by Kleinert.

Under a surface transformation $\tilde{S} = S + dS$, we write

$$\boldsymbol{\delta}(\boldsymbol{x}, \tilde{S}) = \int_{\tilde{S}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, d\boldsymbol{S}' = \int_{S} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, d\boldsymbol{S}' + \int_{dS} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, d\boldsymbol{S}' = \tag{5.23}$$

$$\int_{S} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, d\boldsymbol{S}' + \int_{V} \boldsymbol{\nabla} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\, d\boldsymbol{x}'^{3} = \boldsymbol{\delta}(\boldsymbol{x}, S) + \boldsymbol{\nabla}\delta(\boldsymbol{x}, V),$$

where the second to last equivalence is a consequence of Stokes' theorem and V is the volume bounded by the two surfaces. Hence

$$\boldsymbol{\nabla} \times \boldsymbol{\delta}(\boldsymbol{x}, \tilde{S}) = \boldsymbol{\nabla} \times \boldsymbol{\delta}(\boldsymbol{x}, S). \tag{5.24}$$

Therefore, equation (5.21) is invariant under change of the cut-surface S and then it can be seen as a gauge theory as well as the well known gauge symmetry when summing a gradient of a single-valued function to the velocity field. However, the two gauge theories are different in meaning. Moreover, even the velocity $\mathbf{u}(\boldsymbol{x})$ is invariant under change of surface. This is because, under change of S, the phase $\chi(\boldsymbol{x}, S)$, that is related to the solid angle swept by the defect line, changes as the solid angle changes under this transformation:

$$\chi(\boldsymbol{x}, \tilde{S}) = \int_{\tilde{S}} \frac{\boldsymbol{R'}}{R'^3} \cdot d\boldsymbol{S'} = \int_{S} \frac{\boldsymbol{R'}}{R'^3} \cdot d\boldsymbol{S'} + \int_{dS} \frac{\boldsymbol{R'}}{R'^3} \cdot d\boldsymbol{S'} = \tag{5.25}$$

$$\chi(\boldsymbol{x}, S) - \int_{dS} \boldsymbol{\nabla} \left(\frac{1}{R'}\right) \cdot d\boldsymbol{S'} = \chi(\boldsymbol{x}, S) - \int_{V} \nabla^2 \left(\frac{1}{R'}\right) \cdot d^3\boldsymbol{x}'.$$

Since $\nabla^2 \left(\frac{1}{R'}\right) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')$ then from the last term we obtain

$$\chi(\boldsymbol{x}, \tilde{S}) = \chi(\boldsymbol{x}, S) - \delta(\boldsymbol{x}, V). \tag{5.26}$$

Therefore, under a change of surface S, the change in the gradient of phase $\boldsymbol{\nabla}\chi(\boldsymbol{x}, S)$ and of $\boldsymbol{\delta}(\boldsymbol{x}, S)$ compensate and the velocity field does not depend of the choice of S. Therefore this theory can be seen as a double gauge theory.

The production of curl by a gradient is due to the cut surface, present only if the phase is globally defined and multi-valued. Kleinert's theory has a clear, natural explanation. To understand this physically let us think of BEC as a classical fluid, that at time $t = 0$ is stationary and irrotational (without vortices). In this case all the streamlines are orthogonal to infinite, parallel surfaces at each point of space for which the normal vector is the velocity field $\mathbf{u}$. This parallel surfaces are iso-potential surfaces. Indeed, without vorticity we have $\mathbf{u} = \boldsymbol{\nabla}\chi$, where χ is the velocity potential. In this case it is a single-valued potential and thus it is curl free. Imagine now that at a certain instant of time, abruptly, we put an infinitesimally thin, bi-dimensional boundary layer in the fluid, parallel to the streamlines. Then, under a shear force on that surface, the speed will change producing rotation of the stream lines and thus an almost singular vorticity field. Conversely, given a singular vorticity field, we can always think of it as due to a very thin surface layer present in the fluid. The presence of the vortex is independent on the choice of the surface layer. Following this parallelism, we may attribute the role of the boundary layer to the cut surface S and, consequently, the delta function would play the role of the extra velocity acquired by fluid particles to close the curved streamlines once we remove the layer.

One can manage a solid angle or, similarly, a phase of a BEC as a multi-valued function depending on the vortex line, without fixing any cut-surface S [45]. In this case the function

$$\chi = \chi(\boldsymbol{x}, \gamma) \tag{5.27}$$

is not divided into a discontinuous, simple-valued function depending on S plus a delta function term defined on the cut-surface, but must be regarded with multi-valued analysis methods. However we note this function no more satisfies Schwartz integrability condition of commuting, second partial derivatives:

$$(\partial_i \partial_j - \partial_j \partial_i)\, \chi \neq 0. \tag{5.28}$$

This property comes from the non-vanishing curl of the gradient of χ. Indeed, if we multiply both the sides of the equation for ϵ_{kij} we get exactly the

condition $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\chi \neq 0$.

Chapter 6

Twist, writhe, self-linking and helicity

In this chapter we will review some basic geometric and topological concepts used in topological fluid dynamics and that are fundamental for what follows.

6.1 Twist

A single, classical vortex with a small, finite internal structure can be formed by many thin, vorticity lines. These lines can rotate many times around a central line. Any of these lines is very close to the central one and form, with it, the two edges of what we call a *ribbon*. The rate of rotation of any of these lines around the central line is called *twist*. Given a central vortex line γ with tangent vector $\hat{T}(s)$, a ribbon $\mathcal{R}$ is formed from γ by tracing the distance between γ and a curve infinitesimally close to it. The distance vector is infinitesimal and we call it $\epsilon\hat{U}(s)$, where $\epsilon << 1$ and $\hat{U}(s) \cdot \hat{T}(s) = 0$, $\forall s$. We call $\hat{U}$ the *frame* of the ribbon $\mathcal{R}$. Then we have the following

Definition 6.1.1. Given a curve γ with vector $\boldsymbol{x}(s)$, a ribbon relative to a frame $\hat{U}$ is a couple $\mathcal{R}(\gamma, \hat{U})$ such that the first edge of $\mathcal{R}$ is γ and the second is a curve γ^* defined by the vector $\boldsymbol{x}^*(s) := \boldsymbol{x}(s) + \hat{U}(s)$.

Then we can recall the definition of twist for a classical vortex :

Definition 6.1.2. Given a curve γ and a ribbon $\mathcal{R}(\gamma, \hat{U})$ the twist of the ribbon relative to the frame $\hat{U}$ is

$$Tw(\gamma, \hat{U}) = \frac{1}{2\pi} \int_\gamma \left(\hat{U} \times \frac{d}{d\,s} \hat{U} \right) \cdot \hat{T}\; d\,s. \tag{6.1}$$

Twist admits decomposition in terms of normalized total torsion T and intrinsic twist $\mathcal{N}$ [25], i.e.

$$Tw = \frac{1}{2\pi} \int_\gamma \tau(s)\, \mathrm{d}s + \frac{[\Theta]_\gamma}{2\pi} = T + \mathbf{U}\ , \tag{6.2}$$

where $\tau(s)$ is the local torsion of γ, $[\Theta]_\gamma$ the total rotation angle of $\hat{U}$ around γ and $[\Theta]_\gamma / 2\pi = \mathcal{N}$ the number of full rotations. If γ is a straight line of length L, uniform twist is given by the uniform rotation of $\hat{U}$ around γ with $\mathcal{N}/L = m/L$ the number of full rotations per unit length (see Figure 8.2a). Twist represents a global geometric property of the ribbon.

6.2 Writhe

Sometimes, independently from the presence of twisted line, it is possible that the central line is coiled. The amount of coiling of the central line around itself is quantified by the *writhe*. This is defined as the linking number of the central line with itself, that is the linking number of two curves γ_1 and γ_2 when they are infinitesimally close to each other forming the central curve γ as the limiting curve.

Definition 6.2.1. If we describe the two infinitesimally curves as the same curve γ but with two different parametrizations $\boldsymbol{x}(s)$ and $\boldsymbol{x}(t)$, then, given the vector $\boldsymbol{r} := \boldsymbol{x}(s) - \boldsymbol{x}(t)$, the writhe of γ is defined as

$$Wr(\gamma) = \int_\gamma \int_\gamma \frac{\boldsymbol{r}}{r^3} \cdot (d\,\boldsymbol{x}(s) \times d\,\boldsymbol{x}(t)). \tag{6.3}$$

We note that while Tw depends also on the frame, Wr is a property of the central line itself. Also writhe is a global geometric property of γ. Thus, for classical singular vortices, writhe is well defined, whereas twist is not, because if the vortex is very thin it has no internal structure.

6.3 Self-linking number and Călugăreanu's theorem

Consider a single vortex Then it is possible to define the linking number of the vortex with itself. We can give the following

Definition 6.3.1. The *self-linking number*, Sl, of a vortex is defined by letting one of the two curves in (3.15) tend to the other.

We underline the fact that Sl is a topological invariant of the vortex γ and measure the degree of knottedness of a vortex [26].
Then, if we call Sl the self-linking of γ, we can recall the following ([25])

Theorem 1 (Călugareănu)**.** $Sl = Wr + Tw$

6.4 Helicity

An important physical quantity is kinetic helicity. Given a vorticity field $\boldsymbol{\omega}$ and the velocity field $\mathbf{u}$ such that $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$, we give the following

Definition 6.4.1. Kinetic helicity $\mathcal{H}$ is the quantity defined by

$$\mathcal{H} = \int \boldsymbol{\omega} \cdot \mathbf{u} \, d^3\boldsymbol{x}. \tag{6.4}$$

For singular vortices the helicity have the following decomposition in terms of linking number and self linking number between vortices:

$$\mathcal{H} = \sum_{i}^{m} (\Gamma_i)^2 Sl_i + \sum_{i \neq j}^{m} \Gamma_i \Gamma_j Lk_{ij}. \tag{6.5}$$

An important fact of BECs is that the helicity vanishes: $\mathcal{H} = 0$ [9]. Therefore, given m vortices γ_i each one with circulation Γ_i, their self-linkings Sl_i and linking numbers Lk_{ij} are connected by a fundamental relation (see for instance [9]):

$$\sum_{i}^{m} (\Gamma_i)^2 Sl_i + \sum_{i \neq j}^{m} \Gamma_i \Gamma_j Lk_{ij} = 0. \tag{6.6}$$

6.5 Writhe as a geometric phase

In [56] it is demonstrated that Wr is proportional to the solid angle spanned by the tangent vector placed at the origin of a unit sphere.

As we have already seen, the geometric phase is a phase acquired by a system when its Hamiltonian depends on parameters that vary in time and that end up to the initial value after a period. The variation of these parameters form a closed curve in a parameter space and the geometric phase is proportional to the solid angle Ω swept out by the curve in this parameter space. Imagine now that, instead of time, the evolution is within the three-dimensional space. In this case we have a closed curve γ in configuration space that represents the evolution in time, and the parameter vector that changes in space is just the tangent vector $\hat{T}$ to γ. The parameter space is S^2. The solid angle swept out by the curve generated by $\hat{T}$ on the unit sphere, called *tangent indicatrix*, coincides with the writhe Wr of the curve $mod\, 2\pi$. Thus the writhe of γ can be seen as the geometric phase acquired by a system when its motion depends on the parameter $\hat{T}$ and the spatial evolution of $\hat{T}$ is such that it forms a non-zero solid angle on the unit sphere As an example we can think of a precession motion of the classical spin of a particle around $\hat{T}$ while $\hat{T}$ itself is precessing around a fixed axis (for example the z-axis). In the next section we will rewrite Wr in a different way, underlying its relation with the solid angle of the curve γ. We will follow [22]. Then we will manipulate it in a new way to find an expression for the writhe of γ that will be useful at the end of this work to demonstrate the natural transition from twist phase to writhe in quantum vortices, and hence their natural coiling, in case of local θ_{tw}.

Let us write down the writhe term recalling first the definition of solid angle as we have already seen in the specific section.

$$\Omega(\boldsymbol{x}, \gamma) = \int_{\gamma} \frac{\hat{r}_{\infty} \cdot (\hat{r} \times d\hat{r})}{1 + (\hat{r}_{\infty} \cdot \hat{r})} \qquad mod\, 4\pi. \tag{6.7}$$

Now, consider the closed curve spanned by the tangent of γ. This curve is formed by pushing at infinity any element of γ along the straight line defined

by its tangent and is called *tangent indicatrix* $\gamma_{\hat{T}}$. The writhe of the curve γ is thus the solid angle subtended by $\gamma_{\hat{T}}$. Instead of going to infinity we can project $\gamma_{\hat{T}}$ on an observation sphere. Therefore, a formula for the writhe is directly derived by substituting $\hat{T}$ instead of $\hat{r}$ in (6.7). Let us fix the auxiliary vector $\hat{r}_\infty$. The authors demonstrate that

$$\Omega(\gamma_{\hat{T}}) = \int_{\gamma_{\hat{T}}} \frac{\hat{r}_\infty \cdot \left(\hat{T} \times d\hat{T}\right)}{1 + \left(\hat{r}_\infty \cdot \hat{T}\right)} = 2\pi \left(1 + Wr(\gamma)\right) \qquad mod\, 4\pi. \tag{6.8}$$

Note that the authors arrive at the same conclusion of Füller ([56], eq. 6.1). Since now the curve is that spanned by $\hat{T}$ and the tangent does not depend on the observation point, this reflects the fact that writhe depends only on the curve γ. Moreover we note that as the solid angle does not depend on the fixed, unit vector $\hat{r}_\infty$ the same holds for writhe. Therefore we can always fix it to be the z-axis, $\hat{r}_\infty = \hat{z}$:

$$\Omega(\gamma_{\hat{T}}) = \int_{\gamma_{\hat{T}}} \frac{\hat{z} \cdot \left(\hat{T} \times d\hat{T}\right)}{1 + \left(\hat{z} \cdot \hat{T}\right)} = 2\pi \left(1 + Wr(\gamma)\right) \qquad mod\, 4\pi. \tag{6.9}$$

In [22] it is demonstrated that the solid angle of a space curve γ seen from a point $\boldsymbol{x}$ on the unit sphere is given by the self-linking of the curve minus the total geodesic curvature of the curve on the observation sphere (from the Gauss-Bonnet theorem):

$$\Omega(\boldsymbol{x}, \gamma) = 2\pi \left(1 + Wr\right) + \int_\gamma \frac{\left(\hat{r} \cdot \hat{T}\right)\left[\hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)\right]}{1 - \left(\hat{r} \cdot \hat{T}\right)^2} - \int_\gamma \frac{\hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)}{1 - \left(\hat{r} \cdot \hat{T}\right)^2}. \tag{6.10}$$

The middle part in the rhs of the equation is $2\pi\, Tw$ and, recalling the Călugăreanu theorem, $2\pi\, Sl = 2\pi \left(1 + Wr\right) + 2\pi\, Tw \quad mod\, 4\pi$. The last term on the rhs is the total geodesic curvature.

Therefore ([22])

$$\Omega(\boldsymbol{x}, \gamma) = 2\pi \left(1 + Wr\right) - \int_\gamma \frac{\hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)}{1 + \left(\hat{r} \cdot \hat{T}\right)}, \tag{6.11}$$

where the quantity $-\int_\gamma \frac{\hat{r}\cdot\left(\hat{T}\times d\hat{T}\right)}{1+\left(\hat{r}\cdot\hat{T}\right)}$ is the twist (Tw) minus the total geodesic curvature of γ.

Now we demonstrate the equivalence of the middle term in (6.10) with twist as we have defined it. This statement (only stated in [22] but not demonstrated) is proved by decomposing the curve $\hat{r}$ into a tangential and normal part:

$$\hat{r} = \left(\hat{r}\cdot\hat{T}\right)\hat{T} + \left(\hat{r}\cdot\hat{U}\right)\hat{U}, \tag{6.12}$$

where $\hat{U}$ is a frame orthogonal everywhere to $\hat{T}$, and where the normalization of $\hat{U}$ leads to $\left(\hat{r}\cdot\hat{U}\right)^2 = 1 - \left(\hat{r}\cdot\hat{T}\right)^2$. Then, we have

$$\hat{U}(\boldsymbol{x}) = \frac{\hat{r} - \left(\hat{r}\cdot\hat{T}\right)\hat{T}}{\sqrt{1-\left(\hat{r}\cdot\hat{T}\right)^2}}. \tag{6.13}$$

Now, given a frame $\hat{U}$, the standard definition of the twist of a ribbon $(\gamma, \hat{U})$ is:

$$Tw(\gamma,\hat{U}) = \frac{1}{2\pi}\int_\gamma \left(\hat{U}\times\frac{d}{d\,s}\hat{U}\right)\cdot\hat{T}\,d\,s. \tag{6.14}$$

If we start from this definition and compute the vector product in (6.14) using (6.13) we obtain:

$$\hat{U}\times\frac{d}{d\,s}\hat{U} = \frac{1}{1-\left(\hat{r}\cdot\hat{T}\right)^2}\left[\hat{r}-\left(\hat{r}\cdot\hat{T}\right)\hat{T}\right]\times\left[\frac{d}{d\,s}\hat{r} - \frac{d}{d\,s}\left(\hat{r}\cdot\hat{T}\right)\hat{T} - \left(\hat{r}\cdot\hat{T}\right)\frac{d}{d\,s}\hat{T}\right] \tag{6.15}$$

$$+\frac{1}{[1-\left(\hat{r}\cdot\hat{T}\right)^2]^2}\left[\hat{r}-\left(\hat{r}\cdot\hat{T}\right)\hat{T}\right]\times\left[\left(\hat{r}-\left(\hat{r}\cdot\hat{T}\right)\hat{T}\right)\left(\hat{r}\cdot\hat{T}\right)\frac{d}{d\,s}\left(\hat{r}\cdot\hat{T}\right)\right]$$

$$=\frac{1}{1-\left(\hat{r}\cdot\hat{T}\right)^2}\left[\hat{r}\times\frac{d}{d\,s}\hat{r} - \frac{d}{d\,s}\left(\hat{r}\cdot\hat{T}\right)\hat{r}\times\hat{T} - \left(\hat{r}\cdot\hat{T}\right)\hat{r}\times\frac{d}{d\,s}\hat{T} -\right.$$

$$\left.\left(\hat{r}\cdot\hat{T}\right)\hat{T}\times\frac{d}{d\,s}\hat{r} + \left(\hat{r}\cdot\hat{T}\right)^2\hat{T}\times\frac{d}{d\,s}\hat{T}\right].$$

Taking account that $\frac{d}{d\,s}\hat{r} = \hat{T}$, after some simple algebra one obtains

$$\left(\hat{U} \times \frac{d}{d\,s}\hat{U}\right) \cdot \hat{T} = \frac{\left(\hat{r} \cdot \hat{T}\right)\, \hat{r} \cdot \left(\hat{T} \times \frac{d}{d\,s}\hat{T}\right)}{1 - \left(\hat{r} \cdot \hat{T}\right)^2}. \tag{6.16}$$

Therefore, integrating one obtains the searched form for the twist as stated in [22]:

$$Tw = \frac{1}{2\pi}\int_{\gamma} \frac{\left(\hat{r} \cdot \hat{T}\right)\, \hat{r} \cdot \left(\hat{T} \times \frac{d}{d\,s}\hat{T}\right)}{1 - \left(\hat{r} \cdot \hat{T}\right)^2}\, d\,s = \frac{1}{2\pi}\int_{\gamma_{\hat{T}}} \frac{\left(\hat{r} \cdot \hat{T}\right)\, \hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)}{1 - \left(\hat{r} \cdot \hat{T}\right)^2}. \tag{6.17}$$

Hence the two formulations for the twist are equivalent.

Now we want to rewrite the writhe term as expressed in [22] decomposing the writhe as a difference of angles of rotation. This notation will be useful at the end of the thesis to demonstrate the natural transition of twist phase into writhe for quantum vortices perturbed with localized twist phase. This decomposition serves also as a new point of view for the writhe in terms of angles and underline a (skew)-symmetry for the writhe. To do this we will proceed by steps.

Above we recalled the decomposition of this integral quantity in (6.10):

$$-\int_{\gamma_{\hat{T}}} \frac{\hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)}{1 + \left(\hat{r} \cdot \hat{T}\right)} = \int_{\gamma_{\hat{T}}} \frac{\left(\hat{r} \cdot \hat{T}\right)\left[\hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)\right]}{1 - \left(\hat{r} \cdot \hat{T}\right)^2} - \int_{\gamma_{\hat{T}}} \frac{\hat{r} \cdot \left(\hat{T} \times d\hat{T}\right)}{1 - \left(\hat{r} \cdot \hat{T}\right)^2}. \tag{6.18}$$

Hence this quantity can be decomposed into a twist part, that can be viewed as the total angle of rotation of the frame $\hat{U}$ around $\hat{T}$ in a frame co-moving with the curve (i.e. with respect to a Fermi-Walker-transported frame; see section 7.2 below for the definition of Fermi-Walker transport). For classical systems call this angle $\theta_{\hat{U}}(\hat{T})$. We now demonstrate that the second quantity can be viewed as the total angle of rotation of $\hat{T}$ around $\hat{r}$.

Recall the Euler's equations for a precessing system of axes, in particular now the variation in space of $\hat{T}$ due to its rotation around $\hat{r}$

$$\frac{d}{d\,s}\hat{T} = \boldsymbol{\omega} \times \hat{T}. \tag{6.19}$$

Now vector multiply both sides of (6.19) for $\hat{T}$ and use the vector identity $\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$. We obtain

$$\hat{T}\times\frac{d}{d\,s}\hat{T}=\boldsymbol{\omega}-\left(\boldsymbol{\omega}\cdot\hat{T}\right)\hat{T}. \tag{6.20}$$

Hence, the component of the angular velocity of $\hat{T}$ along $\hat{z}$, $\omega_r:=\boldsymbol{\omega}\cdot\hat{r}$ is

$$\omega_r=\frac{\hat{r}\cdot\left(\hat{T}\times d\hat{T}\right)}{1-\left(\hat{r}\cdot\hat{T}\right)^2}. \tag{6.21}$$

The infinitesimal angle of rotation around the $\hat{r}$-axis is $\omega_r\,d\,s$. Therefore the total angle of rotation of $\hat{T}$ around $\hat{r}$ is

$$\theta_{\hat{T}}(\hat{r})=\int_{\gamma_{\hat{T}}}\frac{\hat{r}\cdot\left(\hat{T}\times d\hat{T}\right)}{1-\left(\hat{r}\cdot\hat{T}\right)^2}. \tag{6.22}$$

This means that the integral quantity of the rhd of (6.10) is the difference of two angles:

$$-\int_{\gamma_{\hat{T}}}\frac{\hat{r}\cdot\left(\hat{T}\times d\hat{T}\right)}{1+\left(\hat{r}\cdot\hat{T}\right)}=\theta_{\hat{U}}(\hat{T})-\theta_{\hat{T}}(\hat{r}). \tag{6.23}$$

Now, we note that the integral in (6.8) is formally equivalent to the second term on the RHS of (6.11), where the equivalence holds substituting $\hat{r}$ with $\hat{z}$, with the main difference that $\hat{z}$ is fixed whereas $\hat{r}$ can change. Therefore, the same decomposition we have used can be done for the writhe (6.8) expressing the writhe, in a "twist" part plus a "geodesic curvature" part.

Given $\hat{z}$, define the associated frame $\hat{z}_\perp$ orthogonal to $\hat{T}$:

$$\hat{z}_\perp=\frac{\hat{z}-\left(\hat{z}\cdot\hat{T}\right)\hat{T}}{\sqrt{1-\left(\hat{z}\cdot\hat{T}\right)^2}}. \tag{6.24}$$

Now, since $\hat{z}$ is fixed then $\frac{d}{d\,s}\hat{z}=0$. Instead $\frac{d}{d\,s}\hat{z}_\perp\neq 0$. Then we can define the twist of $\hat{z}_\perp$ as

$$\int\frac{\left(\hat{z}\cdot\hat{T}\right)\left[\hat{z}\cdot\left(\hat{T}\times d\hat{T}\right)\right]}{1-\left(\hat{z}\cdot\hat{T}\right)^2}. \tag{6.25}$$

Hence we have

$$2\pi\,(1+Wr)=\int\frac{\hat{z}\cdot\left(\hat{T}\times d\hat{T}\right)}{1-\left(\hat{z}\cdot\hat{T}\right)^2}-\int\frac{\left(\hat{z}\cdot\hat{T}\right)\left[\hat{z}\cdot\left(\hat{T}\times d\hat{T}\right)\right]}{1-\left(\hat{z}\cdot\hat{T}\right)^2}. \tag{6.26}$$

Therefore the solid angle spanned by $\hat{T}$ can be decomposed as a difference between the angle of rotation with which $\hat{T}$ rotates around $\hat{z}$ minus that with which the normal component of $\hat{z}$, $\hat{z}_\perp$, rotates around $\hat{T}$:

$$2\pi\,(1+Wr) = \theta_{\hat{T}}(\hat{z}_\perp) - \theta_{\hat{z}_\perp}(\hat{T}). \tag{6.27}$$

The interpretation of writhe is thus the relative angle spanned by the relative rotation of $\hat{T}$ around $\hat{z}$ and of $\hat{z}$ around $\hat{T}$.

It is worth noting that, in a certain sense, the expression on the RHS of equation (6.27) is antisymmetric with respect to the interchange between $\hat{T}$ and $\hat{z}$. The second angle is the twist of $\hat{z}_\perp$ around $\hat{T}$ seen as an axis of rotation. We can do the same fixing $\hat{z}$ as a rotation axis and consider the frame $\hat{T}_\perp$ which is the (normalized) component of $\hat{T}$ perpendicular to $\hat{z}$ in the decomposition

$$\hat{T} = \left(\hat{T}\cdot\hat{z}\right)\hat{z} + \left(\hat{T}\cdot\hat{T}_\perp\right)\hat{T}_\perp. \tag{6.28}$$

Substituting in the expression for $\theta_{\hat{z}}(\hat{T})$ we arrive at

$$\theta_{\hat{z}}(\hat{T}) = \int \left(\hat{T}_\perp \times \frac{d}{d\,s}\hat{T}_\perp\right)\cdot\hat{z}\,d\,s. \tag{6.29}$$

This is the twist of the frame $\hat{T}_\perp$ around $\hat{z}$ seen as a rotation axis. Now the writhe part is made explicitly anti-symmetric:

$$2\pi\,(1+Wr) = \int\left[\left(\hat{T}_\perp \times \frac{d}{d\,s}\hat{T}_\perp\right)\cdot\hat{z} - \left(\hat{z}_\perp \times \frac{d}{d\,s}\hat{z}_\perp\right)\cdot\hat{T}\right] d\,s = \theta_{\hat{T}}(\hat{z}_\perp) - \theta_{\hat{z}_\perp}(\hat{T}_\perp). \tag{6.30}$$

Chapter 7

Twist phase

7.1 Generalization of twist to quantum vortices: twist phase

We note that, while for quantum singular vortices the linking number is well defined, twist is not, because a quantum vortex has no internal structure. We thus need to generalize the concept of twist for a quantum vortex. Since vortices in BECs are singularities of phase, they have no structure. It would seem impossible to define a twist for them. Anyway, they posses an internal, unobservable structure given by the phase of the condensate. Now we propose a generalization of twist for vortices in BECs that is in line with that given in [9]. However we underline other important contributions to the study of twist in optics and condensed matter physics: [57], [58].

Given a vortex γ we can define isophase surfaces, i.e. orientable surfaces of constant phase, that foliate the entire condensate domain. They are also called Seifert surfaces of γ. Since vortex lines are phase defects, in presence of a vortex γ any isophase surface is bounded by γ. Close to the vortex line any isophase surface S is approximated by a plane that is spanned by the tangent $\hat{T}$ and by a (normalized) vector $\hat{U}$, that is orthogonal to $\hat{T}$ and tangent to S at every point of γ. We call $\hat{U}$ a *natural frame* of γ. Any different isophase defines a different natural frame $\hat{U}$.

Now, imagine an isophase surface as a fabric surface with γ the boundary of

this surface. If it is unperturbed it remains flat, also along the boundary. Any perturbation bends locally or globally the fabric surface, creating creases. Moreover, a perturbation can have a privileged direction. Now imagine to perturb the whole fabric in such a way there are creases throughout the surface and that, at least near the boundary, the perturbation is sufficiently large that a portion of the fabric twirls around it going along the boundary. A twist associated with γ produces a very similar situation: it is just a phase perturbation [54], with distortion of the isophase surface, that, close to the vortex grows along the vortex [9]. Close to the vortex line, the amount of the perturbation is indicated by a rotation of the frame $\hat{U}$ that now measures the local rotation of the isophase surface around $\hat{T}$ going along γ with respect to the unperturbed situation. When there is a twist perturbation, $\hat{U}$ is no more parallel transported along the curve. Therefore, close to the curve we get the classical definition of twist, so that now we can define twist even far from the vortex. Since a rotation of the isophase changes the phase along the vortex line, then the perturbed total phase χ has a non-zero component of $\boldsymbol{\nabla}\chi$ along γ. We call the phase field that perturbs any isophase surface a *twist phase* $\theta_{tw}(\boldsymbol{x}, t)$. The total (perturbed) phase of the condensate is the sum of the unperturbed phase and the twist phase. Let us call the perturbed phase χ and the unperturbed phase θ. We have:

Definition 7.1.1 (Twist phase)**.** Given a vortex γ, a *twist phase* is a phase field $\theta_{tw}(\boldsymbol{x}, t)$ such that:

1. the new, perturbed phase χ can be written as the sum of the unperturbed phase θ and θ_{tw}: $\chi(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) + \theta_{tw}(\boldsymbol{x}, t)$;

2. close to γ there must be a non-zero, longitudinal, gradient component of the phase $\boldsymbol{\nabla}\chi \cdot \hat{t}$ such that $\boldsymbol{\nabla}\chi \cdot \hat{t} = \boldsymbol{\nabla}\theta_{tw} \cdot \hat{t} \neq 0$.

Thus, a twist perturbation produces an extra velocity of the condensate $\boldsymbol{\nabla}\theta_{tw}$ that near the vortex is longitudinal to the defect line γ. For the correspondence principle with the classical twist number, in a tubular neighbor

of the vortex γ, twist phase must be

$$\theta_{\rm tw}(s) = \int_0^s \left(\hat{U} \times \frac{d\hat{U}}{d\bar{s}}\right) \cdot \hat{T}\, d\bar{s}\ , \tag{7.1}$$

where $\bar{s}$ a dummy variable. Total twist is given by the cumulative rotation of the ribbon spanwise unit vector $\hat{U}$ from some origin $s = 0$ to L. Here for simplicity we assume γ to be closed and inextensible, with total twist number given by $Tw = (2\pi)^{-1}\theta_{\rm tw}(L)$. In presence of stretching the definition can be easily adapted to include functional dependence of s on time.

An experiment to inject twist phase onto a vortex ring using Gauss-Laguerre beams and geometric phase has been proposed in [54] and recalled in this thesis.

The main aspect of injecting twist on a quantum vortex is that immediately after the perturbation, a second, central vortex can form by changing the linking number of the system (as shown numerically in [12] and demonstrated theoretically in [54], and discussed in [50]). This is a new, quantum effect, not observable for classical vortices and it will be analyzed in details in the following of this thesis. In a context of quantum turbulence it is of high interest to study the dynamics of twisted quantum vortices to see the implications of this on the system and on the cascade of reconnections [55]. Since any phase of a closed defect is multi-valued, to demonstrate theoretically this secondary vortex production we need to apply a mathematical theory to easily manage global, multi-valued fields.

The physical effect of twist is made explicit by considering a stationary, straight defect γ in isolation represented by the straight ribbon of Figure 7.1(a), and the velocity interpretation of the phase gradient $\boldsymbol{\nabla}\chi = \mathbf{u}$, decomposed locally in cylindrical polar components (u_r, u_θ, u_ξ). If the isophase surface $\bar{\chi}$ is *not* twisted it simply coincides with the coordinate plane $\theta =$ constant, with gradient $\mathbf{u}_\theta = (\Gamma/2\pi r)\hat{\mathbf{e}}_\theta$ given by the classical rectilinear vortex solution. In this case the family of isophase surfaces hinged on a straight axis foliate the ambient space by a fan of θ–planes (forming the so-called "open-book decomposition"), with streamlines planar circles centred on the

straight axis γ, and lying on concentric cylinders. When local twist is different from zero the isophase surfaces are no longer planar, forming helical surfaces hinged on γ, now with a non-zero axial velocity $\mathbf{u}_\xi$ *along* γ.

It will be observed that an m-phase perturbation acquired by a wave function of the fundamental state implies the generation of wavefront dislocations as an Aharonov-Bohm effect [49]. Hence the superposition of an m-phase on a defect is expected to induce the production of a secondary singularity of strength m [50]. This is exactly what has been observed in [12]: since the wavefronts are related to the action of an azimuthal velocity $\mathbf{u}_\theta$ around γ, similarly the induced twist due to an m-phase perturbation along γ is associated with an axial velocity $\mathbf{u}_\xi$ along the defect; thus, the n-star wavefronts are led to rotate around the original defect and, as we shall demonstrate below, this gives rise to a secondary vortex of strength m (mod 2π). The corresponding additional axial velocity $\mathbf{u}_\xi$ changes the momentum from $\mathbf{P}$ to $\mathbf{P} + \mathbf{u}_\xi$, with the free part of the GPE hamiltonian given (per unit of mass) by $\hat{H}_{\text{free}} = (\boldsymbol{\nabla} + \mathbf{u}_\xi)^2/2$. This is in analogy with the Hamiltonian of an electron under the influence of a magnetic field, where $\mathbf{u}_\xi$ plays the role of the magnetic vector potential $\mathbf{A}$, as in the original Aharonov-Bohm experiment [42].

7.2 Twist and Fermi-Walker transport

Twist condition. Since twist is locally detected by the rotation of $\hat{U}$ around γ, we need to determine the twist condition in full generality, i.e. in a non-rotating, non-inertial frame. This condition is provided by Fermi-Walker (FW) transport [51, 52]. For a generic triad $\{\hat{T}, \hat{P}, \hat{Q}\}$ ($\hat{Q} = \hat{T} \times \hat{P}$) on $\mathbf{X}^*$, FW-transport of $\hat{P}$ is governed by the equation

$$\frac{\mathrm{D}_{\mathrm{FW}}\hat{P}}{\mathrm{D}s} = \frac{\mathrm{d}\hat{P}}{\mathrm{d}s} - \left(\hat{P}\cdot\hat{T}\right)\frac{\mathrm{d}\hat{T}}{\mathrm{d}s} + \left(\hat{P}\cdot\frac{\mathrm{d}\hat{T}}{\mathrm{d}s}\right)\hat{T}\,. \tag{7.2}$$

Thus (see Figure 7.1c):

Definition 7.2.1 (Zero-twist condition)**.** The unit vector $\hat{P}$ does not rotate

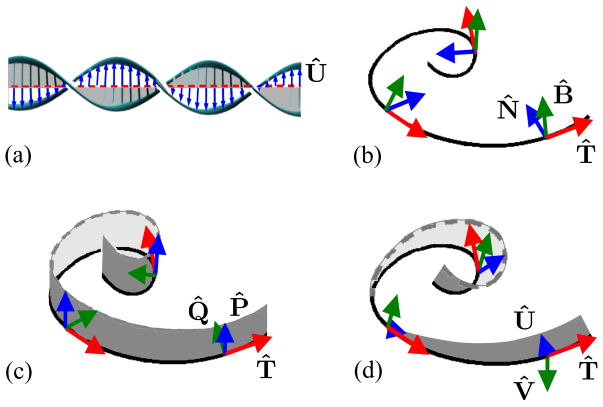

Figure 7.1: (a) Uniform twist visualized by the uniform rotation of the ribbon spanwise unit vector $\hat{U}$ around the ribbon straight axis (color online). (b) Frenet frame $\{\hat{T}, \hat{N}, \hat{B}\}$ on a base space curve (thick, black line); (c) parallel transport frame $\{\hat{T}, \hat{P}, \hat{Q}\}$; (d) phase twist frame $\{\hat{T}, \hat{U}, \hat{V}\}$ (adapted from [54]).

along γ if and only if it is Fermi-Walker-transported along γ, that is

$$\frac{\mathrm{D}_{\mathrm{FW}}\hat{P}}{\mathrm{D}s} = 0 \ , \qquad \forall s \in \mathbf{X}^* \ . \tag{7.3}$$

Let us consider the standard Frenet frame $\{\hat{T}, \hat{N}, \hat{B}\}$ with $\hat{N}$ and $\hat{B}$ the usual normal and binormal unit vectors to $\mathbf{X}^*$ (see Figure 7.1b) and apply eq. (7.2) to the ribbon spanwise unit vector $\hat{U}$. Using Frenet-Serret equations, with c curvature of γ, a first computation gives

$$\frac{\mathrm{D}_{\mathrm{FW}}\hat{U}}{\mathrm{D}s} = \frac{\mathrm{d}\hat{U}}{\mathrm{d}s} - c(\hat{B} \times \hat{U}) \ . \tag{7.4}$$

Taking $\hat{U} = \cos\theta(s)\,\hat{N} + \sin\theta(s)\,\hat{B}$ a second computation gives the point-wise twist condition for $\hat{U}$ along γ, i.e.

$$\frac{\mathrm{D}_{\mathrm{FW}}\hat{U}}{\mathrm{D}s} = (\tau + \theta')\hat{\mathbf{e}}_\theta \ , \tag{7.5}$$

where $\theta' = \mathrm{d}\theta/\mathrm{d}s$ and $\hat{\mathbf{e}}_\theta$ the azimuthal unit vector around γ. This means that a ribbon is twisted iff $\hat{U}$ rotates with rotation rate given by the sum of 2 contributions, τ due to the rotation of the Frenet-Serret triad with respect to the FW frame, and θ' given by the rotation of $\hat{U}$ in the Frenet-Serret system. The isophase twist is thus determined by the twist condition (7.5) on $\hat{U}$, made here explicit by the Fermi-Walker derivative.

Since the rotation of $\hat{U}$ is also governed by the equation of rigid body rotation, we have

$$\frac{\mathrm{D}_{\mathrm{FW}}\hat{U}}{\mathrm{D}s} = \boldsymbol{\Omega}_\xi \times \hat{U} = \Omega(\hat{T} \times \hat{U}) \ . \tag{7.6}$$

Now locally — in a FW frame — we can identify the ribbon spanwise unit vector $\hat{U}$ with the portion of the isophase surface close to the nodal line, given (in general) by $\mathrm{e}^{\mathrm{i}\chi}$; thus, from (7.5) and (7.6) we have

$$\tau + \theta' = \Omega = \frac{\mathrm{d}\chi}{\mathrm{d}s} = \boldsymbol{\nabla}\chi \cdot \hat{T} \ , \tag{7.7}$$

that identifies the rotation rate of the isophase (through $\hat{U}$) with its tangential component along the nodal line. The velocity field interpretation of the phase twist rate given by Hänninen *et al.* (see their eqs. (14-15) in [53]) is thus proved by Madelung transform.

Chapter 8

Dynamics of twisted quantum defects

In the preceding chapters we have defined the concept of twist and have generalized it to Bose-Einstein condensates defining the concept of twist phase. The novelty of this work is to consider a vortex defect in presence of twist and to study its dynamics. We will demonstrate the implications of phase twist on the stability and the dynamics of defects in BECs. Moreover, recent numerical simulations [12] and theoretical work [54] demonstrate that, under certain circumstances, new defects can be produced by pure injection of phase twist on existing defects. As pointed out by [54] this observation has interesting potential applications in science and technology. In order to understand and exploit details associated with twist energy and vortex dynamics we derive the correct set of governing equations for defects in the presence of twist. Since the ambient domain is multiply-connected we must consider appropriate correction to the standard governing equations due to multi-valued potential, and this is done by applying the defect gauge theory developed in [45] and reviewed above.

8.1 Modified Gross-Pitaevskii equation for twisted state

Let us consider the superposition of phase twist $\theta_{\rm tw}$ on an existing defect. In the following we shall consider the space-time evolution of γ, so that $\mathbf{X} = \mathbf{X}(s,t)$ and $\theta_{\rm tw} = \theta_{\rm tw}(s,t)$ on γ, so that in general $\theta_{\rm tw} = \theta_{\rm tw}(\mathbf{x},t)$. Several experimental techniques to inject twist locally and to get it spread in the bulk of the condensate can be used, including the method proposed by [54](see sec. V, B) based on the exploitation of the Berry phase [41]. Twist thus produced on the defect gets then instantly distributed in the condensate because of the contextual nature of the governing wave function that characterizes the system [59].

If ψ_0 denotes the fundamental state, the superposed twist state ψ_{tw} is defined by the following transformation:

$$\psi_0 \to \psi_{tw} = \mathrm{e}^{\mathrm{i}\theta_{\rm tw}(\mathbf{x},t)}\psi_0 \ . \tag{8.1}$$

Since the GPE is not invariant under local phase transformations, in the twist superposition, ψ_{tw} does not evolve under the same GPE. We will determine the modified equation. From (8.1) we have

$$\partial_t\psi_{tw} = (\partial_t \mathrm{e}^{\mathrm{i}\theta_{\rm tw}})\psi_0 + (\partial_t\psi_0)\mathrm{e}^{\mathrm{i}\theta_{\rm tw}} \ . \tag{8.2}$$

Substituting the GPE (2.4) into the right-hand side term above and after re-arranging terms, we have

$$\partial_t\psi_{tw} = \mathrm{i}(\partial_t\theta_{tw})\psi_{tw} + \frac{\mathrm{i}}{2}(\boldsymbol{\nabla}^2\psi_0)\mathrm{e}^{\mathrm{i}\theta_{\rm tw}} + \frac{\mathrm{i}}{2}(1-|\psi_{tw}|^2)\psi_{tw} \ , \tag{8.3}$$

where we have taken $|\psi_0| = |\psi_{tw}|$.

Using the vector identity $\mathrm{e}^{\mathrm{i}\theta_{\rm tw}}\boldsymbol{\nabla}^2\psi_0 = \boldsymbol{\nabla}\cdot(\mathrm{e}^{\mathrm{i}\theta_{\rm tw}}\boldsymbol{\nabla}\psi_0) - (\boldsymbol{\nabla}\mathrm{e}^{\mathrm{i}\theta_{\rm tw}})\cdot\boldsymbol{\nabla}\psi_0$ and substituting (8.1), the Laplacian term becomes

$$\mathrm{e}^{\mathrm{i}\theta_{\rm tw}}\boldsymbol{\nabla}^2\psi_0 = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi_{tw} - \mathrm{i}\boldsymbol{\nabla}\theta_{\rm tw}\,\psi_{tw}) - \mathrm{i}\boldsymbol{\nabla}\theta_{\rm tw}\cdot\mathrm{e}^{\mathrm{i}\theta_{\rm tw}}\boldsymbol{\nabla}\psi_0. \tag{8.4}$$

Considering that

$$\mathrm{e}^{\mathrm{i}\theta_{\rm tw}}\boldsymbol{\nabla}\psi_0 = \boldsymbol{\nabla}\psi_{tw} - \mathrm{i}\boldsymbol{\nabla}\theta_{\rm tw}\psi_{tw}, \tag{8.5}$$

applying vector calculus to solve the bracket we have

$$\mathrm{e}^{\mathrm{i}\theta_{\mathrm{tw}}}\boldsymbol{\nabla}^2\psi_0 = \nabla^2\psi_{tw} - 2\mathrm{i}\boldsymbol{\nabla}\theta_{\mathrm{tw}}\cdot\boldsymbol{\nabla}\psi_{tw} - \mathrm{i}\nabla^2\theta_{\mathrm{tw}}\,\psi_{tw} - |\boldsymbol{\nabla}\theta_{\mathrm{tw}}|^2\psi_{tw}. \tag{8.6}$$

After some straightforward algebra we arrive at

$$\partial_t\psi_{tw} = \mathrm{i}(\partial_t\theta_{\mathrm{tw}})\psi_{tw} + \frac{\mathrm{i}}{2}(\boldsymbol{\nabla} - \mathrm{i}\boldsymbol{\nabla}\theta_{\mathrm{tw}})^2\psi_{tw} + \frac{1}{2}\boldsymbol{\nabla}\theta_{\mathrm{tw}}\cdot\boldsymbol{\nabla}\psi_{tw} + \frac{\mathrm{i}}{2}\left(1 - |\psi_{tw}|^2\right)\psi_{tw}\ . \tag{8.7}$$

The correct evolution equation is a modified form of the standard Gross-Pitaevskii equation, given by

$$\partial_t\psi_{tw} = \frac{\mathrm{i}}{2}\tilde{\boldsymbol{\nabla}}^2\psi_{tw} + \frac{\mathrm{i}}{2}\left(1 - |\psi_{tw}|^2\right)\psi_{tw} + \mathrm{i}(\partial_t\theta_{\mathrm{tw}})\psi_{tw} + \frac{1}{2}\boldsymbol{\nabla}\theta_{\mathrm{tw}}\cdot\boldsymbol{\nabla}\psi_{tw} \qquad \text{(mGPE)}, \tag{8.8}$$

where $\tilde{\boldsymbol{\nabla}} = \boldsymbol{\nabla} - \mathrm{i}\boldsymbol{\nabla}\theta_{\mathrm{tw}}$; the mGPE is still a Gross-Pitaevskii type of equation, with an extra interaction term given by phase twist and a flux term proportional to the gradient of θ_{tw} along γ (for explicit computation of this flux and its interpretation in terms of axial fluid flow see again [54] and Sec. 8.5 below).

8.2 Hamiltonian and energy

Before considering the actual vortex dynamics, few considerations about Hamiltonian and energy of the twisted state are in order. The single-particle Hamiltonian [17] (see, for instance, p. 111) associated with the mGPE above is given by

$$H_{\mathrm{tw}} = \frac{1}{2}\tilde{\boldsymbol{p}}^2 - \frac{1}{2}(1 - |\psi_{tw}|^2) - V_{\mathrm{tw}}\ , \tag{8.9}$$

where $\tilde{\boldsymbol{p}} = \mathbf{p} - \boldsymbol{\nabla}\theta_{\mathrm{tw}}$ is the canonical momentum of the twisted state, $\mathbf{p} = -\mathrm{i}\boldsymbol{\nabla}$ is the momentum operator, and $V_{\mathrm{tw}} = \partial_t\theta_{\mathrm{tw}} + (1/2)\boldsymbol{\nabla}\theta_{\mathrm{tw}}\cdot\mathbf{p}$ is the twist potential. We have

Lemma 1. *The Hamiltonian H_{tw} associated with the mGPE (8.8) is non-Hermitian, that is $H_{\mathrm{tw}}^\dagger \neq H_{\mathrm{tw}}$, where † denotes the adjoint operator.*

Proof. The quadratic terms of $\tilde{\boldsymbol{p}}^2$ are Hermitian since both $\mathbf{p}$ and $\boldsymbol{\nabla}\theta_{\mathrm{tw}}$ (proportional to the identity, and function of $\mathbf{x}$) are individually Hermitian;

the self-interaction terms are also Hermitian (with pre-factor $(1-|\psi_{tw}|^2)$, a real number); the remaining part, however, is not Hermitian:

$$(\boldsymbol{\nabla}\theta_{\rm tw}\cdot\mathbf{p})^\dagger = \mathbf{p}^\dagger\cdot\boldsymbol{\nabla}\theta_{\rm tw}^\dagger = \mathbf{p}\cdot\boldsymbol{\nabla}\theta_{\rm tw}\ ;$$

$$(\boldsymbol{\nabla}\theta_{\rm tw}\cdot\mathbf{p})^\dagger\psi_{tw} = (\mathbf{p}\cdot\boldsymbol{\nabla}\theta_{\rm tw})\psi_{tw} = -\mathrm{i}\,\boldsymbol{\nabla}^2\theta_{\rm tw}\,\psi_{tw}\ ;$$

$$(\boldsymbol{\nabla}\theta_{\rm tw}\cdot\mathbf{p})\,\psi_{tw} = -\mathrm{i}\,\boldsymbol{\nabla}\theta_{\rm tw}\cdot\boldsymbol{\nabla}\psi_{tw} \neq (\boldsymbol{\nabla}\theta_{\rm tw}\cdot\mathbf{p})^\dagger\psi_{tw}\ .$$

Hence $H_{\rm tw}^\dagger \neq H_{\rm tw}$. ■

Properties of non-Hermitian Hamiltonians have been widely investigated in both classical and quantum contexts [60], [61], [62],[63], mainly when physical systems manifest loss and gain of energy. The implications for twisted state become clear when we compute the energy. This is given by the energy of the unperturbed state (2.6) plus the energy given by the twist. Since the latter is linear in the order parameter ψ_{tw}, it can be obtained from the expectation value of V_{tw} and of the kinetic part that depends on θ_{tw}, that is by extrapolating the twist part of $\langle\psi_{tw}|H_{tw}|\psi_{tw}\rangle/\langle\psi_{tw}|\psi_{tw}\rangle$ (normalizing on the total number of particles present). Thus the total energy is given by

$$E_{\rm tw} = \int\Big[\Big(\frac{1}{2}|\boldsymbol{\nabla}\theta_{tw}|^2 - \partial_t\theta_{\rm tw} + \mathrm{i}\boldsymbol{\nabla}\theta_{\rm tw}\cdot\boldsymbol{\nabla}\psi_{tw} + \frac{\mathrm{i}}{2}\nabla^2\theta_{tw}\Big)\,|\psi_{tw}|^2 \qquad (8.10)$$

$$+\frac{1}{2}|\boldsymbol{\nabla}\psi_{tw}|^2 - \frac{1}{2}|\psi_{tw}|^2 + \frac{1}{4}|\psi_{tw}|^4\Big]dV\ ,$$

where dV denotes the volume element of the condensate. We see that the expectation value of V_{tw} (first term in brackets) has, in this case, an imaginary part. Upon application of the Madelung transform $\psi_{tw} = \sqrt{\rho}\exp(\mathrm{i}\chi_{tw})$ on $\mathrm{i}\boldsymbol{\nabla}\theta_{\rm tw}\cdot\boldsymbol{\nabla}\psi_{tw}$ and considering that $\boldsymbol{\nabla}\theta_{tw}\cdot\boldsymbol{\nabla}\rho = 0$ because the gradients are orthogonal in a neighbor of the vortex, we obtain

$$E_{\rm tw} = \int\Big[\Big(\frac{1}{2}|\boldsymbol{\nabla}\theta_{tw}|^2\,|\psi_{tw}|^2 - \partial_t\theta_{\rm tw}\,|\psi_{tw}|^2 - \boldsymbol{\nabla}\theta_{\rm tw}\cdot\boldsymbol{\nabla}\chi_{tw}\,|\psi_{tw}|^2 \qquad (8.11)$$

$$+\frac{1}{2}|\boldsymbol{\nabla}\psi_{tw}|^2 - \frac{1}{2}|\psi_{tw}|^2 + \frac{1}{4}|\psi_{tw}|^4\Big) + \frac{\mathrm{i}}{2}\nabla^2\theta_{tw}\,|\psi_{tw}|^2\Big]dV\ ,$$

where this time the real and imaginary parts have been separated. It is the imaginary term above that makes the Hamiltonian non-Hermitian; moreover $E_{\rm tw}$ depends on time through $\theta_{\rm tw}$ and $\partial_t\theta_{\rm tw}$. The non-Hermiticity of $H_{\rm tw}$ provides an alternative proof that a twisted state in isolation is indeed unstable and undergoes a sort of gain and loss process. Confirmation of this comes from linear perturbation of the mGPE as is described in detail in the next section. It is clear, however, that the information on the Hamiltonian H_{tw} is still incomplete because it is unclear what $\partial_t\theta_{tw}$ is, then we must find an equation that describes the time evolution of θ_{tw} in order to have a complete information of the dynamics of the twisted state. In section (8.5) we will find such an equation in order to give a complete set of differential equations describing the dynamics of the state.

8.3 Linear stability analysis of the twisted vortex state

Consider a twisted vortex state ψ_{tw}. Perturb a portion of the BEC close to the vortex with small, linear perturbations ψ_{twp}. Since the scale of variation of the twist phase along the vortex is much larger than that of the perturbed neighbor, then a separation of variables is needed. In the neighbor of each point a different dispersion relation is needed that takes account of the different value of θ_{tw} at that site.
The perturbed state is then given by

$$\tilde{\psi} = \psi_{tw} + \psi_{twp} = \psi_{tw} + |\psi_{twp}|\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\nu\, t)} \qquad |\psi_{twp}| \ll |\psi_{tw}| \ , \tag{8.12}$$

where ψ_{tw} is the unperturbed state and $|\psi_{twp}| =$constant, $\mathbf{k}$ wave vector, ν perturbation frequency. By substituting $\tilde{\psi}$ into (8.8), using the fact that ψ_{tw}

is a solution for (8.8) and retaining only first-order terms in ψ_{twp} we obtain

$$\partial_t\psi_{twp} = \mathrm{i}\,(\partial_t\theta_{\mathrm{tw}})\psi_{twp} + \frac{\mathrm{i}}{2}(\boldsymbol{\nabla} - \mathrm{i}\boldsymbol{\nabla}\theta_{\mathrm{tw}})^2\psi_{twp} + \frac{1}{2}\boldsymbol{\nabla}\theta_{\mathrm{tw}}\cdot\boldsymbol{\nabla}\psi_{twp} \tag{8.13}$$

$$+ \frac{\mathrm{i}}{2}\left(1 - |\psi_{tw}|^2\right)\psi_{twp} - \mathrm{i}\,|\psi_{tw}|^2\mathrm{Re}\left(\frac{\psi_{twp}}{\psi_{tw}}\right)\psi_{tw}$$

$$. \tag{8.14}$$

Since we want to perturb the system in a region close to the vortex we consider also $\psi_{tw} \ll 1$. With this further approximation, retaining only first-order terms and remembering that ρ, $|\psi_{1p}|$, $\mathbf{k}$ and ν does not depend on space and time, the dispersion relation becomes:

$$\nu = \frac{1}{2}\Big[\left(|\mathbf{k}|^2 - 2\boldsymbol{\nabla}\theta_{\mathrm{tw}}\cdot\mathbf{k} + |\boldsymbol{\nabla}\theta_{\mathrm{tw}}|^2 - 1 - \partial_t\theta_{\mathrm{tw}}\right) + \mathrm{i}\nabla^2\theta_{\mathrm{tw}}\Big]\,. \tag{8.15}$$

Considering the imaginary part of the frequency, we can state the following:

– $\nabla^2\theta_{\mathrm{tw}} < 0$: we have damped oscillations with $\tilde{\psi} \to \psi_0$ as $t \to \infty$;

– $\nabla^2\theta_{\mathrm{tw}} = 0$: oscillatory terms survive and the system is stable;

– $\nabla^2\theta_{\mathrm{tw}} > 0$: we have instability with $|\psi| \to \infty$ as $t \to \infty$.

Non-Hermiticity of the total Hamiltonian is translated into the complexity of total energy $E_{\mathrm{tw}} = \mathrm{Re}(E_{\mathrm{tw}}) + \mathrm{i}\,\mathrm{Im}(E_{\mathrm{tw}})$. Since the energy is

$$E_{\mathrm{tw}} = \int\left[\left(\frac{1}{2}|\boldsymbol{\nabla}\psi_{tw}|^2 + U(\psi_{tw})\right) + \frac{i}{2}\,|\psi_{tw}|^2\nabla^2\theta_{tw}\right]d\,V, \tag{8.16}$$

where $U(\psi_{tw}) = -\frac{1}{2}|\psi_{tw}|^2 + \frac{1}{4}|\psi_{tw}|^4 - \partial_t\theta_{tw}|\psi_{tw}|^2 + \frac{1}{2}|\boldsymbol{\nabla}\theta_{tw}|^2\,|\psi_{tw}|^2 - \boldsymbol{\nabla}\theta_{tw}\cdot\boldsymbol{\nabla}\chi_{tw}\,|\psi_{tw}|^2$ we note again the dependence on $\nabla^2\theta_{\mathrm{tw}}$. Since $|\psi_{tw}|^2 = \rho$ it is worth noting that while the instability due to the formation of Kelvin waves with growing amplitude remains whatever ρ is, the total energy of the condensate in a twisted state is complex only in a small region $0 < \rho < 1$ close to the vortex, called the *healing-length* of the vortex. In summary we have:

i) if $\boldsymbol{\nabla}^2\theta_{\mathrm{tw}} < 0$, then $\mathrm{Im}(E_{\mathrm{tw}}) > 0$. H_{tw} is non-Hermitian with non-conserved (probability) density and phase diffusion due to the flux of $\rho\boldsymbol{\nabla}\theta_{\mathrm{tw}}$ along γ. The system is linearly *stable* under small perturbations;

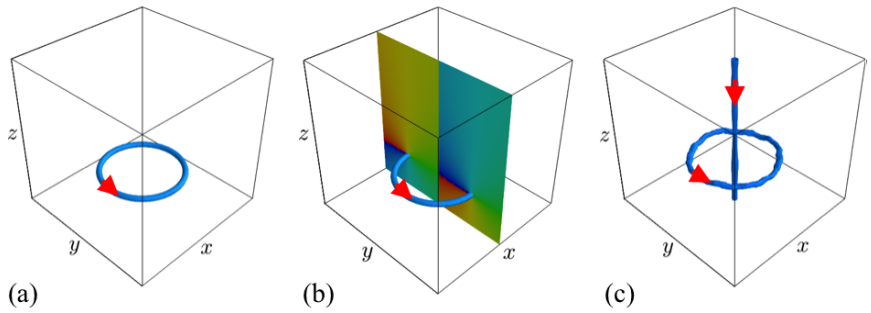

Figure 8.1: (a) Initial condition given by a planar vortex ring visualized by the iso-density tubular surface $\rho = 0.1$. The arrow indicates the vorticity direction. (b) Twist $Tw = 1$ is superposed (say at $t = 0$) by prescribing a full rotation of the isophase $\bar{\chi}$ (as described in detail in Section 8.3.1), shown by the phase contour in the (y, z)-plane (color online). (c) The presence of twist induces the instantaneous production of a new, central defect, here shown at a later time [adapted from[12]]. Note the corrugation of the iso-density surface that reflects perturbation of the density profile (taken from [54]).

ii) if $\boldsymbol{\nabla}^2\theta_{\rm tw} > 0$, then $\mathrm{Im}(E_{\rm tw}) < 0$, then $H_{\rm tw}$ is non-Hermitian with non-conserved (probability) density and phase diffusion due to the flux of $\rho\boldsymbol{\nabla}\theta_{\rm tw}$ along γ. The system is linearly *unstable* under small perturbations and produces growing-amplitude Kelvin waves.

iii) if $\boldsymbol{\nabla}^2\theta_{\rm tw} = 0$, then $E_{\rm tw}$ is real, the Hamiltonian is Hermitian and the system is linearly *stable* under small perturbations.

As mentioned by [64], pointing out the possible role of topology in relation to stability aspects, in the vortex ring case of Figure 8.1 we have an explicit demonstration that twist instability is indeed what triggers topological changes and the corresponding production of a new defect. The process can be seen as an energy relaxation mechanism in response to twist superposition.

8.3.1 Vortex ring case

The simple case of uniform phase twist superposed on a vortex ring γ_0 of radius R_0 provides an explicit example. In the zero-twist case the vortex propagates steadily in the medium without change of shape (see Figure 8.1a), with conserved energy $E_0 = E(\theta_{\rm tw} = 0)$ [17]. Suppose now that at some time (say $t = 0$) we superpose instantly uniform twist $\theta_{\rm tw} = w\alpha$, where w denotes the winding number of $\hat{U}$ around γ_0 and α is the azimuth angle. In cylindrical coordinates (r, α, z) centered on the ring, we have

$$\boldsymbol{\nabla}\theta_{\rm tw} = \frac{w}{R_0}\hat{\mathbf{e}}_\alpha \ , \qquad \partial_t\theta_{\rm tw} = \boldsymbol{\nabla}\theta_{\rm tw}\cdot\boldsymbol{\nabla}\psi_{tw} = 0 \ , \tag{8.17}$$

where $\hat{\mathbf{e}}_\alpha$ is the azimuth unit vector. The new energy state is now given by

$$E_{\rm tw} = \int \Big(\frac{1}{2}\Big|\boldsymbol{\nabla}\psi_{tw} - {\rm i}w\frac{w}{R_0}\hat{\mathbf{e}}_\alpha\psi_{tw}\Big|^2 - \frac{1}{2}|\psi_{tw}|^2 + \frac{1}{4}|\psi_{tw}|^4\Big)dV \ . \tag{8.18}$$

Evidently, we have

$$E_{\rm tw} = E_0 + \frac{1}{2}\Big(\frac{w}{R}\Big)^2|\psi_{tw}|^2 \geq E_0 \ , \tag{8.19}$$

where $E_{\rm tw}$ is quadratic in w, and $E_{\rm tw} = E_0$ if and only if $w = 0$. As [54] demonstrated, topological and dynamical arguments show that superposition of twist generates in this case instantaneous production of a new, central defect, as shown in Figure 8.1(c). This aspect will be clarified in the following chapters.

8.4 Hydrodynamic equations in the presence of twist

It is well known [17] that by applying the Madelung transformation to (2.4) the real and imaginary parts of the GPE give rise to the momentum and continuity equation of a fluid-like medium. By following the same procedure, substitution of $\psi_{tw} = \sqrt{\rho}\exp({\rm i}\chi_{tw})$ into the mGPE gives rise to the set of equations

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\theta_{\rm tw}) \ , \tag{8.20}$$

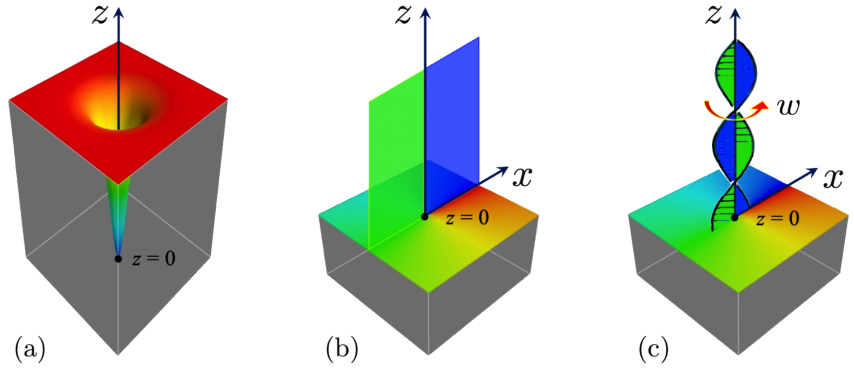

Figure 8.2: Straight defect along the z axis: (a) density profile around the nodal line $\rho = 0$; (b) case of zero phase twist, with phase jump across the positive half-plane (x, z) (evidenced by the blue half-plane online); (c) case of uniform phase twist with winding number w (taken from [54]).

$$\partial_t \chi_{tw} = \frac{1}{2}[1 - \rho - (\mathbf{u} - \boldsymbol{\nabla}\theta_{\rm tw})^2] + Q + \partial_t \theta_{\rm tw} \ , \tag{8.21}$$

where $Q = \boldsymbol{\nabla}^2\sqrt{\rho}/(2\sqrt{\rho})$ is the so-called quantum potential. Equation (8.20) is the new continuity equation, where change in (probability) density is now balanced by diffusion of twist; the gradient of eq. (8.21) generates the momentum equation in hydrodynamic form. When a defect is present this equation must be modified to take into account the multi-valued phase. To implement this correction we follow the defect gauge theory developed by [45], applied by [65] and summarized in the above chapters, to multi-valued potentials. Indeed, in analogy with Helmholtz's decomposition of classical fluid mechanics, we have

$$\mathbf{u} = \mathbf{u}_{\rm I} + \mathbf{u}_{\rm R} = \boldsymbol{\nabla}\chi_{tw} + \mathbf{A} = \boldsymbol{\nabla}\chi_{tw} + A\boldsymbol{\delta}_{\bar{\chi}_{tw}}(\mathbf{x}) \ , \tag{8.22}$$

where $\mathbf{u}_{\rm I}$ is irrotational and $\mathbf{u}_{\rm R} \equiv \mathbf{A}$ denotes the rotational contribution given by the vector potential $\mathbf{A} = A\boldsymbol{\delta}_{\bar{\chi}_{tw}}(\mathbf{x})$ due to the singular distribution of vorticity; $\mathbf{A}$ is directed along the normal to the cut-isophase surface $\bar{\chi}_{tw}$ through which we have the phase jump (see Figure 8.2b, c); we take $A =$ constant and $\boldsymbol{\delta}_{\bar{\chi}_{tw}}(\mathbf{x}) = \int_{\bar{\chi}_{tw}} \delta^{(3)}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}'$, $\forall \mathbf{x}' \in \bar{\chi}_{tw}$ [45].

Following [65] it is convenient to write $\mathbf{A}$ in terms of the 4 components $\{A_\mu\}$, with $\mu = 1, 2, 3$ to denote space and $\mu = 0$ time. In analogy with the electric/magnetic decomposition associated with the Faraday tensor, we

write

$$E_\mu = \partial_0 A_\mu - \partial_\mu A_0 \ . \tag{8.23}$$

Consider now the 4-dimensional velocity field, that in components is given by

$$u_\mu = \frac{{\psi_{tw}}^* \partial_\mu \psi_{tw} - \psi_{tw} \partial_\mu \psi^*_{tw}}{2i \psi^*_{tw} \psi_{tw}} \tag{8.24}$$

(ψ^*_{tw} complex conjugate), where now $u_0 \equiv \partial_0 \chi_{tw} + A_0$. In order to determine the new time derivative of the velocity when the defect is present, we note that

$$\partial_\mu u_0 = \partial_\mu (\partial_0 \chi_{tw} + A_0) = \partial_0 (\partial_\mu \chi_{tw} + A_\mu) + (\partial_\mu A_0 - \partial_0 A_\mu) \ ; \tag{8.25}$$

substituting (8.23) into the last bracket of the equation above, we have $\boldsymbol{\nabla} u_0 = \partial_t \mathbf{u} - \mathbf{E}$, i.e.

$$\partial_t \mathbf{u} = \boldsymbol{\nabla} u_0 + \mathbf{E} \ , \tag{8.26}$$

that gives the correct momentum equation. In another paper [79] similar results have been done relative to the velocity of a defect line, without twist, even though following another way. Another, variational approach for the study of the dynamics of untwisted defects has been explored in [80].

Since $\mathbf{u} = \boldsymbol{\nabla} \chi_{tw} + \mathbf{A}$, to take account of the cut effects on the time component of $\mathbf{u}$ we must replace the term $\partial_t \chi_{tw}$ on the right-hand side of (8.21) with $\partial_t \chi_{tw} + A_0$, i.e. $\partial_t \chi_{tw} \to \partial_t \chi_{tw} + A_0$, so that eq. (8.21) becomes

$$u_0 = \partial_t \chi_{tw} + A_0 = \frac{1}{2}[1 - \rho - (\mathbf{u} - \boldsymbol{\nabla} \theta_{\rm tw})^2] + Q + \partial_t \theta_{\rm tw} \ . \tag{8.27}$$

$\mathbf{E}$ can then be worked out considering the jump of the vector potential $\mathbf{A}$ through the cut-isophase surface $\bar{\chi}_{tw}$. Since defect dynamics is independent of the cut-isosurface, we can always choose the cut (where $\mathbf{A}$ is defined) such that $\chi_{tw} = 0$; for

$$\frac{{\rm Im}(\psi_{tw})}{{\rm Re}(\psi_{tw})} = \tan \chi_{tw} = \tan 0 = 0 \ . \tag{8.28}$$

Hence, ${\rm Im}(\psi_{tw}) = 0$ for all ${\rm Re}(\psi_{tw}) > 0$ (see again Figure 8.2b, c). This means that $\partial_\mu \chi_{tw}$ must have a discontinuity given by [65] $-2\pi\, \Theta[{\rm Re}(\psi_{tw})]\, \partial_\mu \Theta[{\rm Im}(\psi_{tw})]$, so that the jump in the potential $\mathbf{A} = A \boldsymbol{\delta}_{\bar{\chi}_{tw}}(\mathbf{x})$ is given by

$$A_\mu = 2\pi\, \Theta[{\rm Re}(\psi_{tw})]\, \partial_\mu \Theta[{\rm Im}(\psi_{tw})] \ , \tag{8.29}$$

where $\Theta[\cdot]$ is Heaviside's function (for this derivation consider the jump of $\boldsymbol{\delta}_{\bar{\chi}_{tw}}(\mathbf{x})$ across the cut in cylindrical polar coordinates; see [65], eq. 30). Now we compute the field $\mathbf{E}$ explicitly and arrive at the form of $\mathbf{E}$ as written in [65].

From (8.23) we have

$$E_i = 2\pi \left[\partial_0 \left(\Theta[\mathrm{Re}(\psi_{tw})]\, \partial_i \Theta[\mathrm{Im}(\psi_{tw})]\right) - \partial_i \left(\Theta[\mathrm{Re}(\psi_{tw})]\, \partial_0 \Theta[\mathrm{Im}(\psi_{tw})]\right)\right] \tag{8.30}$$

$$\begin{aligned} &= 2\pi \left[\delta(\mathrm{Re}(\psi_{tw}))\delta(\mathrm{Im}(\psi_{tw}))\, (\partial_0 \mathrm{Re}(\psi_{tw})\, \partial_i \mathrm{Im}(\psi_{tw})) - (\partial_i \mathrm{Re}(\psi_{tw})\, \partial_0 \mathrm{Im}(\psi_{tw}))\right] + \\ &\quad \Theta[\mathrm{Re}(\psi_{tw})]\, (\partial_0 \partial_i \mathrm{Im}(\psi_{tw}) - \partial_i \partial_0 \mathrm{Im}(\psi_{tw})) \\ &= 2\pi \delta(\mathrm{Re}(\psi_{tw}))\delta(\mathrm{Im}(\psi_{tw}))\, [(\partial_0 \mathrm{Re}(\psi_{tw})\, \partial_i \mathrm{Im}(\psi_{tw})) - (\partial_i \mathrm{Re}(\psi_{tw})\, \partial_0 \mathrm{Im}(\psi_{tw}))]\,. \end{aligned}$$

Since $\delta(x)\delta(y) = (2\pi)^{-1}\delta(x^2+y^2)$ and since $|\psi_{tw}|^2 = \rho$, then we have

$$E_i = 4\delta(\rho)\, [\partial_0 \mathrm{Re}(\psi_{tw})\, \partial_i \mathrm{Im}(\psi_{tw}) - \partial_i (\mathrm{Re}(\psi_{tw}))\, \partial_0 \mathrm{Im}(\psi_{tw})].\,, \tag{8.31}$$

Writing $\mathrm{Re}(\psi_{tw}) = \frac{1}{2}\,(\psi_{tw} + \psi^*_{tw})$ and $\mathrm{Im}(\psi_{tw}) = \frac{1}{2\,\mathrm{i}}\,(\psi_{tw} - \psi^*_{tw})$ we have

$$\begin{aligned} E_i = 4\delta(\rho)\, \Bigg[&\frac{1}{2}\,(\partial_0 \psi_{tw} + \partial_0 \psi^*_{tw})\, \frac{1}{2\,\mathrm{i}}\,(\partial_i \psi_{tw} - \partial_i \psi^*_{tw}) \\ &-\frac{1}{2}\,(\partial_i \psi_{tw} + \partial_i \psi^*_{tw})\, \frac{1}{2\,\mathrm{i}}\,(\partial_0 \psi_{tw} - \partial_0 \psi^*_{tw})\Bigg]\,. \end{aligned} \tag{8.32}$$

After some algebra we obtain

$$E_i = 2\delta(\rho)\, (\partial_i \psi_{tw} \partial_0 \psi^*_{tw} - \partial_0 \psi_{tw} \partial_i \psi^*_{tw})\,. \tag{8.33}$$

Restoring the gradient and the time derivative and inserting the Madelung transformation also for the conjugate of the wave-function, $\psi^*_{tw} = \sqrt{\rho}\mathrm{e}^{-\mathrm{i}\,\chi_{\mathrm{tw}}}$,

we have

$$\mathbf{E} = 2\delta(\rho)\left[\left(\frac{\boldsymbol{\nabla}\rho}{2\sqrt{\rho}} + \mathrm{i}\,\sqrt{\rho}\,(\boldsymbol{\nabla}\chi_{\mathrm{tw}} + \mathbf{A})\right)\left(\frac{\partial_t\rho}{2\sqrt{\rho}} - \mathrm{i}\,\sqrt{\rho}\,(\partial_{\mathrm{t}}\chi_{\mathrm{tw}} + \mathrm{A}_0)\right) - \right. \tag{8.34}$$

$$\left.\left(\frac{\boldsymbol{\nabla}\rho}{2\sqrt{\rho}} - \mathrm{i}\,\sqrt{\rho}\,(\boldsymbol{\nabla}\chi_{\mathrm{tw}} + \mathbf{A})\right)\left(\frac{\partial_t\rho}{2\sqrt{\rho}} + \mathrm{i}\,\sqrt{\rho}\,(\partial_{\mathrm{t}}\chi_{\mathrm{tw}} + \mathrm{A}_0)\right)\right]$$

$$= 2\delta(\rho)\left[\partial_t\rho\,(\boldsymbol{\nabla}\chi_{tw} + \mathbf{A}) - (\partial_t\chi_{tw} + A_0)\,\boldsymbol{\nabla}\rho\right].$$

Substituting (8.20), (8.21) into the equation and taking account of (8.27), after some algebra we arrive at the form of $\mathbf{E}$ as written in [65] eq. 33,

$$\mathbf{E} = -2\delta(\rho)\left[\boldsymbol{\nabla}\cdot(\rho\mathbf{u} - \rho\boldsymbol{\nabla}\theta_{\mathrm{tw}})\,\mathbf{u} + u_0\boldsymbol{\nabla}\rho\right]. \tag{8.35}$$

By substituting (8.35) into (8.26) we have the correct momentum equation, given by

$$\partial_t\mathbf{u} = \boldsymbol{\nabla}u_0 - 2\delta(\rho)\left[\boldsymbol{\nabla}\cdot(\rho\mathbf{u} - \rho\boldsymbol{\nabla}\theta_{\mathrm{tw}})\,\mathbf{u} + u_0\boldsymbol{\nabla}\rho\right]. \tag{8.36}$$

Note the presence of the δ-function on the right-hand side of the equation above, that justifies the correction to the standard momentum equation (8.21) due to the presence of the defect at $\rho = 0$.

8.5 Twist kinematics and vortex dynamics

First let us consider twist kinematics when γ evolves in time. For this we consider the standard intrinsic reference frame on γ given by the Frenet triad $\{\hat{T}, \hat{N}, \hat{B}\}$, where $\hat{N}$ and $\hat{B}$ denote principal unit normal and binormal vectors to γ, respectively. The time evolution of θ_{tw} is given by two contributions: one, referred to as 'dynamical phase' [40], is due to the Lagrangian rotation of the ribbon unit vector $\hat{U}$ around γ; the other, referred to as 'geometric phase', is due to the evolution of $\mathbf{X}$ (hence $\hat{T}$) in space. The evolution of twist in moving filaments (or ribbons) is derived by [66] by adding together

these two contributions; this is given by

$$\partial_t \theta_{\rm tw} = \int_0^s \left[\boldsymbol{\nabla} \chi_{tw} \cdot \hat{T} + c\, (\partial_t \hat{T})_B \right] d\bar{s} \ , \tag{8.37}$$

where $\boldsymbol{\nabla} \chi_{tw} \cdot \hat{T}$ can be interpreted as a hydrodynamical axial flow that generates the rotation of $\hat{U}$ along γ (see [54]for details) and represents the dynamical phase contribution (cf. eq. 2 of [66]). Here we assumed γ to be inextensible, a plausible assumption in the immediacy of the transient stage of twist superposition. The other term $c\,(\partial_t \hat{T})_B \equiv c\, \partial_t \hat{T} \cdot \hat{B}$ denotes the binormal component of the time derivative of $\hat{T}$ (where by definition $\partial_t \hat{T} \equiv \partial_{ts} \mathbf{X} = \partial_{st} \mathbf{X}$ arc-length derivative of the velocity of γ), and it is responsible for the geometric phase associated with the motion of γ in space (for its derivation consider the intrinsic kinematics of a curve in space;[66] eq. 3). For our purpose it is convenient to re-write everything in terms of $\hat{T}$; from the first Frenet-Serret equations we have $c\, \hat{B} = c\, \hat{T} \times \hat{N} = \hat{T} \times \partial_s \hat{T}$, so that

$$\partial_t \theta_{\rm tw} = \int_0^s \left[\boldsymbol{\nabla} \chi_{tw} \cdot \hat{T} + \partial_t \hat{T} \cdot (\hat{T} \times \partial_s \hat{T}) \right] d\bar{s} \ . \tag{8.38}$$

Vorticity is defined by $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$; since for quantum fluids vorticity is assumed to be a singular distribution of vorticity on nodal lines, using the notation of Kleinert (5.21) we can simply take (after appropriate re-scaling) $\boldsymbol{\omega} = \delta(\mathbf{x} - \mathbf{X}) \hat{T}$, so that $\boldsymbol{\omega} \propto \hat{T}$, function of s and t as well. Equation (8.38) can thus be re-written in terms of $\boldsymbol{\omega}$ by

$$\partial_t \theta_{\rm tw} = \int_0^s \left[\boldsymbol{\nabla} \chi_{tw} \cdot \boldsymbol{\omega} + \partial_t \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \partial_s \boldsymbol{\omega}) \right] d\bar{s} \ . \tag{8.39}$$

Vortex dynamics is governed by the curl of the momentum equation (8.36). By taking the curl of both sides of (8.36), applying the vector identity

$$\boldsymbol{\nabla} \times (f \boldsymbol{a}) = \boldsymbol{\nabla} f \times \boldsymbol{a} + f\, \boldsymbol{\nabla} \times \boldsymbol{a} \tag{8.40}$$

for any scalar function f and vector field $\boldsymbol{a}$, and taking account of the fact that the vector field

$$\boldsymbol{\nabla} \delta(\rho) = \delta' \boldsymbol{\nabla} \rho, \tag{8.41}$$

where δ' is the distributional derivative of the Dirac's delta function, is parallel to $\boldsymbol{\nabla} \rho$, after some repeated application of these facts we arrive at an

expression for the time derivative of the vorticity field:

$$\begin{aligned}\partial_t\boldsymbol{\omega} = \; & - \; 2\boldsymbol{\nabla}\delta(\rho)\times[\boldsymbol{\nabla}\cdot(\rho\mathbf{u}-\rho\boldsymbol{\nabla}\theta_{\rm tw})\,\mathbf{u}] - 2\delta(\rho)\,[\boldsymbol{\nabla}\cdot(\rho\mathbf{u}-\rho\boldsymbol{\nabla}\theta_{\rm tw})\,\boldsymbol{\omega} \\ & + \; \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot(\rho\mathbf{u}))\times\mathbf{u} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\theta_{\rm tw}))\times\mathbf{u} + \boldsymbol{\nabla}u_0\times\boldsymbol{\nabla}\rho] \;\; . \end{aligned} \tag{8.42}$$

The set of governing equations is now complete: we have 3 fundamental variables, ρ, u_0 and $\theta_{\rm tw}$, whose evolution is governed by the following equations:

$$\begin{cases} \partial_t\rho = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\theta_{\rm tw}) - \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) \;, \\ u_0 = \dfrac{1}{2}[1-\rho-(\mathbf{u}-\boldsymbol{\nabla}\theta_{\rm tw})^2] + \dfrac{1}{2\sqrt{\rho}}\boldsymbol{\nabla}^2\sqrt{\rho} + \partial_t\theta_{\rm tw} \;, \\ \partial_t\theta_{\rm tw} = \displaystyle\int_0^s \big[\boldsymbol{\nabla}u_0\cdot\boldsymbol{\omega} + \partial_t\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\times\partial_s\boldsymbol{\omega})\big]d\bar{s} \;. \end{cases} \tag{8.43}$$

These equations are then supplemented by the dynamics of the defect itself, given by eq. (8.42). The numerical implementation of these equations will allow us to study in pure hydrodynamic terms the detailed evolution of quantum defects and superposed twist.

8.6 Global and local twist phases

In the previous sections, as studied in [67], it is shown that a twisted vortex can be unstable: a perturbation of a twisted vortex with phase twist θ_{tw} produces Kelvin waves that can destabilize the vortex if $\nabla^2\theta_{tw} > 0$. Moreover, the energy of the system is complex if $\nabla^2\theta_{tw} \neq 0$ as can be easily shown by considering the expression for the energy E in [67] and performing an integration by part to the imaginary part of E:

$$E = \int\left[\left(\frac{1}{2}|\boldsymbol{\nabla}\psi_{tw}|^2 + U(\psi_{tw})\right) + \frac{i}{2}\,|\psi_{tw}|^2\nabla^2\theta_{tw}\right]\,dV, \tag{8.44}$$

The main idea is that the destabilization process ends when the imaginary part of the energy goes to zero while the real part is minimized in order for the system to reach a more stable state. In what follows we will present two particular mechanisms that may occur in different situations to stabilize the system. To do so we have to distinguish two main possibilities for the twist phase injected into the system and give the following

Definition 8.6.1 (Global twist phase)**.** A twist phase is called *global* if its twist effect extends throughout any fixed isophase surface. Equivalently, the phase is global if, for any fixed isophase surface, the gradient of the twist phase near γ is parallel transported along the lines defined initially by the vector field $\hat{U}$.

Definition 8.6.2 (Local twist phase)**.** A twist phase is called *local* if it changes along the vortex only when it is close to it (of the order of the healing length). It follows that a twist phase is local if its gradient has a longitudinal component with respect to the defect γ and if θ_{tw} becomes constant in each other transversal direction.

We note that it is not necessary that the twist phase is exactly zero far from the curve, but it is sufficient that it is constant, because the GPE is invariant under global $U(1)$ gauge transformations, i.e. the system is invariant under constant phase transformations of the order parameter.
In the following sections we will analyze both the cases of global or local twist phase and the different mechanisms the system undergoes to become stable when $\nabla^2\theta_{tw} > 0$.

8.6.1 Stability from self-linking change: production of a secondary vortex.

Let us start with the first scenario. Consider a global twist phase perturbation θ_{tw} with a non-vanishing gradient along $\hat{T}$. This phase is multi-valued and its gradient is not defined at the center of any fixed isophase surface S. Now consider the total phase χ of the twisted vortex state in the BEC. This can be decomposed into the sum of two, usually multi-valued phases, an unperturbed phase θ and θ_{tw}:

$$\chi(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) + \theta_{tw}(\boldsymbol{x}, t), \quad \forall \boldsymbol{x}, t. \tag{8.45}$$

Given this decomposition we can define loci of constant θ_{tw}. Fix as reference phase the unperturbed phase θ. Let us analyze a region close to the vortex.

Given an initial point s_0 on the defect γ, a twisted isophase near the defect becomes a ribbon surface with one edge in γ and described by a given initial frame $\hat{U}_0 = \hat{U}(s_0)$ that locally stays on the unperturbed isophase $\theta = const.$. This means that all the points along $\hat{U}_0$ have the same, initial phase θ. If we imagine to travel along the vortex in a tubular neighborhood, at any point s following the defect we would measure a different, extra phase $\theta_{tw}(s)$. Hence, neighboring points would not stay on the same isophase nor on the same isotwist surface. If we wanted to select only those points that stay on the $\theta_{tw} = 0$ surface, we should select those little bit downward the isophase $\theta = conts.$, having the same initial phase. And so on along the defect. This is why, for a fixed initial phase $\chi(s_0) = \theta$ the surface defined by $\theta_{tw} = 0$ is equivalent to a ribbon with an edge on the defect. Therefore, globally the isotwist surface $\theta_{tw} = 0$ coincides with the twisted isophase surface $\chi = \chi(s_0)$ and near the vortex this is equivalent to the twisted ribbon spanned by $\hat{U}$ with initial condition $\hat{U}_0$. We deduce that the locus $\theta_{tw} = const. = k$ globally is the twisted isophase $\chi = \chi(s_0) + k$, while near the defect it is equivalent to the twisted ribbon spanned by $\hat{U}_k$, that is the frame $\hat{U}$ rotated by an angle k, with initial condition $\hat{U}_{k,0} = \mathcal{R}(k)\hat{U}_0$, where $\mathcal{R}(k)$ is the rotation matrix around $\hat{T}$ of an angle k.

Therefore, for any fixed, reference phase θ, isotwist surfaces are ribbons.

Since, in this case, the twist phase injected into the system is global, all the isotwist surfaces intersect in a central point. However, physics must be invariant from the choice of a particular reference θ, therefore isotwist surfaces intersect at a central point for any θ. Hence, at the center there will form a line γ_{tw}, where the twist phase is not defined. This is a secondary phase defect produced at a central region and linked with the first one. Hence, as demonstrated in [54], a global twist phase changes the linking number of the system. It is worth noting that if we would had fixed an initial phase θ we would have obtained the same result. In this case any second edge of the ribbon surface would have collapsed to a singular point at the center where the phase is not defined. This would happen for every ribbon surface,

creating a twist phase defect line at a central region.

With this in mind and since θ_{tw} is multi-valued and global, we can apply Kleinert's theory [45] for θ_{tw} to demonstrate the secondary vortex production. Since it is multi-valued on the entire Σ, its gradient, that is the velocity induced by the twist phase, call it $\mathbf{u}_{tw}$, is not curl-free and can be decomposed into a curl-free part plus a delta potential defined only on Σ that is not curl-free:

$$\mathbf{u}_{tw} = \boldsymbol{\nabla}\theta_{tw} = \boldsymbol{\nabla}\theta_{tw}(\Sigma) + \mathbf{u}_{tw}(\Sigma), \tag{8.46}$$

$$\mathbf{u}_{tw}(\Sigma) = \Gamma_{tw}\boldsymbol{\delta}(\Sigma), \tag{8.47}$$

where Γ_{tw} is the circulation of the field $\mathbf{u}_{tw}$ around the intersection line γ_{tw} of all the Σ_s surfaces. Both the quantities in the rhs of equation (8.46) depend on the choice of the cut surface Σ. The sum of the delta function is needed because the sum of these two part cannot depend on Σ, since it is the twist velocity $\mathbf{u}_{tw}$. Following Kleinert's theory we can compute the curl of both sides of the equation, taking into account that

$$\boldsymbol{\omega}_{tw} := \boldsymbol{\nabla} \times \mathbf{u}_{tw} = \boldsymbol{\nabla} \times \mathbf{u}_{tw}(\Sigma) = \Gamma_{tw}\boldsymbol{\delta}(\partial\Sigma) = \Gamma_{tw}\boldsymbol{\delta}(\gamma_{tw}). \tag{8.48}$$

This leads to

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\theta_{tw} = \boldsymbol{\omega}_{tw} \neq \mathbf{0}. \tag{8.49}$$

This second, central singularity γ_{tw} has a non-zero flux through the surface S bounded by γ. This has a topological nature, that is it is proportional to a topological phase, because it is independent of the particular choice of γ, but only on how big is the jump in θ_{tw} through Σ traveling around γ. Hence a global twist phase perturbation changes the initial linking number of the system.

The multi-valuedness of θ_{tw} and the fact that it is global are the causes for the production of the secondary vortex. The gradients of the two phases, $\mathbf{u}_{unp.} = \boldsymbol{\nabla}\theta$ and $\mathbf{u}_{tw}$ respectively, have non-zero curls $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_{tw}$. The single-valued parts that produces the non-zero curls are respectively $\mathbf{u}_{unp.}(S)$ and $\mathbf{u}_{tw}(\Sigma)$. Moreover, as for gradients, any time derivative for multi-valued

phases adds two scalar potentials, as already seen in the previous chapter, $u_{unp.\,0}$ and $u_{tw\,0}$ defined on S and on Σ respectively. With these in mind let us consider the equation expressing the time evolution of the phase

$$\partial_t \chi = Q - \frac{1}{2}|\boldsymbol{\nabla}\chi|^2 + \frac{1}{2}(1-\rho), \tag{8.50}$$

where Q is the quantum potential that depends on the condensate density ρ, and modify it by applying Kleinert's theory, here also for θ_{tw}:

$$\partial_t(\theta+\theta_{tw}) + (u_{unp.\,0} + u_{tw\,0}) = Q - \frac{1}{2}|\boldsymbol{\nabla}(\theta+\theta_{tw}) + (\mathbf{u}_{unp.} + \mathbf{u}_{tw})\,|^2 + \frac{1}{2}(1-\rho), \tag{8.51}$$

Now we take the gradient of both sides. Considering that Q is single-valued, even though it blows up on the vortex, we have

$$\boldsymbol{\nabla}\left[\partial_t(\theta+\theta_{tw}) + (u_{unp.\,0} + u_{tw\,0})\right] = \boldsymbol{\nabla}(single - valued\,part). \tag{8.52}$$

Following [67] and [65] we have

$$\boldsymbol{\nabla}\left[\partial_t(\theta+\theta_{tw}) + (u_{unp.\,0} + u_{tw\,0})\right] = \partial_t \boldsymbol{\nabla}\left(\theta+\theta_{tw}\right) + \boldsymbol{E}, \tag{8.53}$$

where $\boldsymbol{E} := \partial_t\left(\mathbf{u}_{unp.} + \mathbf{u}_{tw}\right) - \boldsymbol{\nabla}\left(u_{unp.\,0} + u_{tw\,0}\right)$ and we changed the sign of $\boldsymbol{E}$ with respect to (8.35) taking already account of the minus sign in the expression for $\boldsymbol{E}$. Then, substituting this expression we have

$$\partial_t \boldsymbol{\nabla}\left(\theta+\theta_{tw}\right) + \boldsymbol{E} = \boldsymbol{\nabla}(single - valued\,part). \tag{8.54}$$

Now, taking the curl at both sides gives

$$\partial_t(\boldsymbol{\omega} + \boldsymbol{\omega}_{tw}) = -\boldsymbol{\nabla} \times \boldsymbol{E}, \tag{8.55}$$

that is the vorticity equation for the system. We note how the vector fields $\boldsymbol{\omega}+\boldsymbol{\omega}_{tw}$ and $\boldsymbol{E}$ are similar to magnetic and electric fields respectively, $\mathbf{u}_{unp.}(S)$ and $\mathbf{u}_{tw}(\Sigma)$ are similar to vector potentials defined on S and Σ, and $u_{unp.\,0}(S)$ and $u_{tw\,0}(\Sigma)$ are similar to scalar potentials defined on the same surfaces. Hence, the fields defined on surface cuts of the two multi-valued phases are like electromagnetic fields.

Now, we want to find out how the flux through S of the new singularity $\boldsymbol{\omega}_{tw}$

changes in time. Since $\boldsymbol{\omega} = \omega \hat{T}$, this does not contribute to the flux through any S bounded by γ. Using Stokes' theorem we obtain

$$\partial_t \int \boldsymbol{\omega}_{tw} \cdot d\,S = -\oint \boldsymbol{E} \cdot \hat{T}\, d\,s = \oint \boldsymbol{\nabla}(u_{unp.\,0} + u_{tw\,0}) \cdot \hat{T}\, d\,s - \partial_t \int \mathbf{u}_{tw} \cdot \hat{T}\, d\,s \tag{8.56}$$

$$= -\partial_t \int \boldsymbol{\nabla}\theta_{tw} \cdot \hat{T}\, d\,s,$$

where we used the fact that $u_{unp.\,0}$ and $u_{tw\,0}$ are both single valued, scalar potentials. Therefore, any variation of the gradient of twist is compensated by the formation of the second singularity that balances any instability given by the twist phase perturbation. Since now

$$\nabla^2\theta_{tw} = \nabla^2\theta_{tw}(\Sigma) + \boldsymbol{\nabla} \cdot \mathbf{u}_{tw}, \tag{8.57}$$

the instability of any positive $\nabla^2\theta_{tw}$ is compensated by the formation of a field $\mathbf{u}_{tw}$ that compensates exactly this effect. All the energy is spent in producing the second singularity and this stabilizes the system.

8.7 Production of the new defect

8.7.1 Global twist effects: pure topological explanation of the new defect.

In the preceding sections we demonstrated how an injection of a global twist phase induces the production of a new, central defect. Another proof of this fact is simply based on pure topological arguments. We remind that the GPE helicity (6.4) is always zero and then the sum of all the linking numbers and the self-linking numbers is zero too (6.6).

In the case of a single vortex ring in isolation exemplified by Figure 8.3(a) we have $Lk_{ij} = 0$ and $SL_1 = Wr_1 + Tw_1 = 0$. However, if superposition of twist gives $Tw_1 = +1$ (with pure azimuthal flow), since for a plane circle $Wr_1 = 0$, we have a contradiction with the zero helicity requirement: indeed

from (6.6), we have

$$\mathcal{H}/\Gamma = SL_1 = 0 \neq Wr_1 + Tw_1 = +1 \ .$$

The situation can only be rectified by the presence of a new, secondary defect (a straight vortex line in the centre) with $Wr_2 = 0$ and $Tw_2 = +1$, so that by (6.5) and (6.6) we have

$$\mathcal{H} = 0 = 2Lk_{12} + Tw_1 + Tw_2 = 2(-1) + 1 + 1 = 0 \ . \tag{8.58}$$

It is this topological condition that makes the physical production of the new defect a manifestation of the Aharonov-Bohm effect.
The production of a new defect admit also an hydrodynamics interpretation as is analyzed in the next subsection.

8.7.2 Global twist effects: hydrodynamics interpretation of the new defect

We have seen that injecting twist produces a velocity along the vortex. To fix ideas let us consider a uniform twist phase along a vortex ring. If twist phase is superposed by a uniform twist rate of $2\pi m/L$, using (7.6) we have

$$\frac{\mathrm{D}_{\mathrm{FW}}\hat{U}}{\mathrm{D}s} = \frac{2\pi m}{L}(\hat{T} \times \hat{U}) = \frac{2\pi m}{L}\hat{\mathbf{e}}_\theta \ . \tag{8.59}$$

By equating (7.5) with (8.59), we have a relation between local geometry and uniform twist, i.e. $\tau + \theta' = 2\pi m/L$. Moreover, by equating (7.4) with (8.59), we can determine the standard rotation rate of $\hat{U}$, given by

$$\frac{\mathrm{d}\hat{U}}{\mathrm{d}s} = \frac{2\pi m}{L}\hat{\mathbf{e}}_\theta + c(\hat{B} \times \hat{U}) = \frac{2\pi m}{L}\hat{\mathbf{e}}_\theta - c\cos\theta\,\hat{T} \ . \tag{8.60}$$

Using the Madelung transform again, we can interpret the 2 contributions on the right-hand side above in terms of an azimuthal flow $\tilde{\mathbf{u}}_\theta = (2\pi m/L)\,\hat{\mathbf{e}}_\theta$ around the defect line and an axial flow $\tilde{\mathbf{u}}_\xi = -c\cos\theta\,\hat{T}$ along the line. This proves the hydrodynamic consequence of twist.

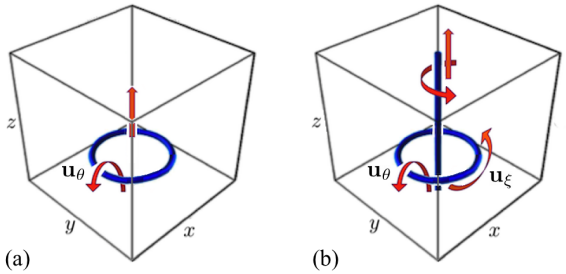

Figure 8.3: Hydrodynamic interpretation of the phase twist on a vortex ring: (a) generation of a secondary, central flow field without singularity by the action of a pure azimuthal velocity $\mathbf{u}_\theta$ on the vortex ring; (b) generation of a new singularity along the ring axis by the combined action of an azimuthal and an axial velocity field $\mathbf{u}_\theta$ and $\mathbf{u}_\xi$ along the ring centreline (taken from [54]).

Production of new defect by twist induction. As an example consider the test case of Figure 8.1(a), where $c = R^{-1}$ is the curvature of the vortex ring. The decomposition (8.60) that corresponds to the simultaneous generation of an azimuthal and an axial flow is essential to understand the formation of a new singularity on the central axis of the ring; using the Biot-Savart law we evidently see how the induction of a pure azimuthal flow $\mathbf{u}_\theta$ (assuming for the moment $\mathbf{u}_\xi = \mathbf{0}$; see Figure 8.3a) can only generate pure axial flow along the central axis, without creating any circulation around it, and hence no new defect. The production of a new defect can only take place when there is also circulation around the z-axis and this, by the Biot-Savart law, can be achieved when $\mathbf{u}_\xi \neq \mathbf{0}$ along the vortex ring as shown in Figure 8.3(b). This is in good agreement with what has been observed in numerical experiments [12].

8.8 Twist induction by phase perturbation

A direct relation between twist effects and phase perturbation is obtained by considering the superposition of a perturbation ψ_p onto the GPE fundamental state ψ_0, i.e. $\psi = \psi_0 + \psi_p$ ($|\psi_p| \ll |\psi_0| \ll 1$). This calculation is very

similar to that done in the linear stability analysis expressed by equation (8.12). However it is a different situation: in the case of equation (8.12) we have considered a perturbation of an already twisted vortex and looked for the dispersion relation. In this case we want to perturb a vortex with an infinitesimal twist perturbation to look for first order consequences of the twist perturbation on the BEC.

In absence of twist, perturbations are known to generate Kelvin waves: indeed, by substituting the new ψ into (2.4) and retaining linear terms in ψ_p, we recover the Schrödinger equation for ψ_p. With reference to the vortex ring case and by taking (R, ϕ, z) cylindrical coordinates centred on the ring central axis at $z = 0$, we have

$$\psi_p = \lambda \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R} - \nu t)} \ , \qquad |\lambda| = \text{constant} \ll 1 \ , \tag{8.61}$$

where ν is the perturbation frequency, $\boldsymbol{k}$ the wave vector, and $\boldsymbol{R}$ the perturbed vector position. The dispersion relation for Kelvin waves is given by

$$\nu = \frac{1}{2}(k^2 - 1) \ , \tag{8.62}$$

from which we can see that the group velocity $\boldsymbol{\nabla}_{\boldsymbol{k}}\nu$ is linear in $\boldsymbol{k}$, where $\boldsymbol{\nabla}_{\boldsymbol{k}}$ indicates the gradient with respect to $\boldsymbol{k}$.

Let us consider the presence of twist. This corresponds to superpose a shifted phase increment on the ground state wave. Let $w = \theta/\phi$ be the winding number given by the ratio of poloidal to toroidal angles. The superposition of total twist through w full rotations on the ring defect (no torsion) generates a new unperturbed state:

$$\psi_0 \longrightarrow \ \psi_0 \, \mathrm{e}^{\mathrm{i}w\phi} \ , \qquad w \in \mathbb{Z} \ . \tag{8.63}$$

Since GPE is not locally phase invariant, we obtain a new equation where the phase twist plays the role of a new potential, i.e.

$$\frac{\partial \psi_0}{\partial t} = \frac{\mathrm{i}}{2}\left(\boldsymbol{\nabla} + \frac{\mathrm{i}w}{R}\hat{\mathbf{e}}_\phi\right)^2 \psi_0 + \frac{\mathrm{i}}{2}(1 - |\psi_0|^2)\psi_0 \ . \tag{8.64}$$

By perturbing this new, fundamental state, we have

$$\psi = \mathrm{e}^{\mathrm{i}w\phi}\psi_0 + \lambda \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R} - \nu t)} \ , \qquad |\lambda| = \text{cst.} \ll 1 \ ; \tag{8.65}$$

and by linearizing we obtain the new equation for ψ_p, given by

$$\frac{\partial \psi_p}{\partial t} = \frac{\mathrm{i}}{2}\left(\boldsymbol{\nabla} + \frac{\mathrm{i}w}{R}\hat{\mathbf{e}}_\phi\right)^2 \psi_p + \frac{\mathrm{i}}{2}\psi_p \ . \tag{8.66}$$

This equation is very similar to that of Aharanov-Bohm [42, 49, 37], where the additional term $(\mathrm{i}w/R)\hat{\mathbf{e}}_\phi$ plays the role of the vector potential created by a singular magnetic field at the origin. Note that $(w/R)\hat{\mathbf{e}}_\phi = (\Gamma/2\pi R)\hat{\mathbf{e}}_\phi$ can indeed be interpreted as the velocity field induced by the central straight vortex. By substituting the plane wave form of ψ_p and considering steady conditions in cylindrical coordinates, we obtain the new dispersion relation

$$\nu = \frac{1}{2}\left(\boldsymbol{k} + \frac{w}{R}\hat{\mathbf{e}}_\phi\right)^2 - \frac{1}{2} \ , \tag{8.67}$$

where now the group velocity depends linearly also on w. Note that the same result can be obtained by direct perturbation of the fundamental state governed by the standard GPE equation (2.4) through a linear perturbation that encapsulates also a phase shift contribution.

8.9 Production of twisted vortex defects in BECs

8.9.1 Production and manipulation of BECs defects

Defects in BECs can be produced by transferring orbital angular momentum. A common method to transfer angular momentum and spin to defects is by using Gauss–Laguerre beams [73]. Polarization of the beam modifies the spin state of the atoms, while orbital angular momentum (OAM) modifies the phase. A Gauss-Laguerre beam in the state LG_p^w is characterized by two integers p and w: $p+1$ denotes the number of nodes of the beam, and w the winding number. So, in general, LG_p^w beams can transfer an azimuthal phase $\mathrm{e}^{\mathrm{i}w\phi}$ to Bose-Einstein condensates (BECs). An example of an LG_0^1 transfer is that given in [6], taking

$$LG_0^1 = \frac{2}{\sqrt{\pi}}\frac{r_0}{\delta_1^2}\mathrm{e}^{-\frac{r_0^2}{\delta_1^2}}\mathrm{e}^{\mathrm{i}\phi} \ , \tag{8.68}$$

where r_0 is the radius of the beam from its center (with singularity at $r_0 = 0$) and $\sqrt{2}\delta_1$ is the peak-to-peak beam diameter. This mode transfers an OAM of $\hbar$ units to each atom of the defect. A singularity is thus created by a Raman transition, exciting the atoms hit by the beam to acquire a higher spin state.

8.9.2 Proposed experiment

To create a vortex ring with phase twist we propose to use a toroidal trap as in [74] and injection of twist by the Berry phase technique used in [41]. First of all the orthogonal quadrupole field for the magnetic trap must be modified in order to produce a variation of the rotation axis of the atoms' angular momentum along the toroidal direction ϕ of the trap (see again [41] for details). This can be achieved by prescribing a field with twist given by a combination of toroidal and poloidal components as in [75]. By doing so we will imprint the generation of a vortex ring defect and, simultaneously, a superposed phase twist given by the rotation axis of the atoms' angular momentum. This rotation will now be subject to the combined effects along the toroidal direction ϕ and the poloidal direction θ, responsible for the phase twist along ϕ. As a result the induced velocity will have a toroidal component along the vortex ring (absent in the case of zero phase twist) that will generate the straight vortex defect at the center of the torus. By perturbing the vortex ring thus generated one can then measure the Kelvin waves propagation as done in [69], and compare the dispersion relation with our eq. (8.67).

An alternative method to generate phase twist is to use LG_p^w beams to induce an OAM on the atoms near the torus nodal line. The defect phase should then acquire a winding number around the nodal line induced by the LG_p^w beam. Again, by measuring the dispersion relation one can get information on the group and phase velocity of the perturbation, and by eq. (8.67) determine the relationship between the wave frequency and the winding number w of the LG_p^w beam. A secondary straight defect of topological charge

proportional to w (in units of circulation) should appear at the center of the vortex ring. Since this will presumably be dynamically unstable [76], we expect its decay to a number of singly-charged defects, orderly self-organized in the central region of the trap. We conjecture that if we interfere a BEC in a rotational "twisted" state with a non-rotational (or counter-rotating) state, the interference is likely to produce a forklike pattern identical to that observed in the standard Aharonov-Bohm effect. In reconstructing the images from the relative interference patterns the two effects will be presumably indistinguishable.

8.10 Stability from writhe generation

8.10.1 Minimization of theta twist

In the previous section we saw the effects of injecting a global twist phase on quantum defects. Now let us consider the case of a local twist phase.
Since from our definition (8.6.2) it is clear that the phase is not singular at the center, we conclude that no second vortex will form. This means that, even if the twist phase remains multi-valued along a tubular neighborhood of the vortex, we have

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\theta_{tw} = \mathbf{0}. \tag{8.69}$$

However the gradient of the twist phase $\boldsymbol{\nabla}\theta_{tw}$ can destabilize the vortex if $\nabla^2\theta_{tw} > 0$. Moreover, a twisted vortex state is unstable and undergoes a sort of gain and loss process, described by the complex energy (8.44), if and only if $\nabla^2\theta_{tw} \neq 0$. Under small perturbations Kelvin waves will form along the vortex and, if $\nabla^2\theta_{tw} > 0$. They will exponentially grow in amplitude because the energy is greater than that of an untwisted one, as shown in details in [67]. Observe that the phase twist can change if the tangent to γ changes its spatial configuration turning into a new $\hat{T}'$ such that, in the new configuration, $\nabla^2\theta_{tw} = 0$. This can happen because a modification of the spatial configuration of the defect changes the total phase χ. Therefore the

presence of a twist perturbation depends on what happens along the defect line. Therefore the twist phase itself depends on the vortex line. Hence, there can exist a perturbation of the defect line such that a change in the total phase compensates or extinguishes the destabilizing twist effects making $\nabla^2\theta_{tw} = 0$.

Let us express the Laplacian of the local twist phase all in terms of the tangent vector $\hat{T}$ and of the normal frame $\hat{U}$. If we describe the gradient operator as the sum of a normal and tangential component, $\boldsymbol{\nabla} = \hat{U}\left(\hat{U}\cdot\boldsymbol{\nabla}\right)+\hat{T}\left(\hat{T}\cdot\boldsymbol{\nabla}\right)$, then we have

$$\boldsymbol{\nabla}\theta_{tw} = \hat{U}\left(\hat{U}\cdot\boldsymbol{\nabla}\theta_{tw}\right) + \hat{T}\left(\hat{T}\cdot\boldsymbol{\nabla}\theta_{tw}\right). \tag{8.70}$$

The Laplacian of θ_{tw} is thus

$$\nabla^2\theta_{tw} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\theta_{tw} = \left[\hat{U}\left(\hat{U}\cdot\boldsymbol{\nabla}\right) + \hat{T}\left(\hat{T}\cdot\boldsymbol{\nabla}\right)\right]\cdot\left[\hat{U}\left(\hat{U}\cdot\boldsymbol{\nabla}\theta_{tw}\right)\right. \tag{8.71}$$

$$\left.+\hat{T}\left(\hat{T}\cdot\boldsymbol{\nabla}\theta_{tw}\right)\right].$$

If we apply the local phase definition, for which $\theta_{tw} \to const.$ rapidly in any direction normal to the defect line, then the expression becomes

$$\nabla^2\theta_{tw} = \hat{T}\left(\hat{T}\cdot\boldsymbol{\nabla}\right)\cdot\left[\hat{T}\left(\hat{T}\cdot\boldsymbol{\nabla}\theta_{tw}\right)\right] = \hat{T}\cdot\boldsymbol{\nabla}\left(\hat{T}\cdot\boldsymbol{\nabla}\theta_{tw}\right). \tag{8.72}$$

Thus the defect must evolve until it reaches a new spatial configuration γ', described by the unit tangent $\hat{T}'$, such that in that configuration $\nabla^2\theta_{tw} = 0$. This means that the longitudinal twist velocity $\mathbf{u}_{tw}$ in the new configuration must be constant along γ' to minimize the imaginary part of the energy. However, in its evolution the vortex must minimize *all* its energy. Here, therefore, we must describe mechanisms the defect is subject to in order to minimize not only the Laplacian gain-loss effects, but also the gradient of θ_{tw}, thus minimizing both the real and imaginary parts of the energy. There are two different mechanisms of evolution of the defect that minimize all the twist effects. We will see that the second one minimizes more efficiently the energy and thus it is considered more natural to happen during the evolution

of the vortex.

The first evolution of $\hat{T}$ that minimizes the twist phase is such that the unperturbed phase of reference θ is re-defined and modified at every point of the new configuration such that the twist phase is everywhere compensated by this new setting of the reference phase. Hence, in this configuration the initial twist is compensated by adding an opposite twist phase to θ at any point. At the end the total phase χ, measured close to the new vortex γ', is everywhere equal to the unperturbed, initial phase θ. The net effect is that in the new configuration γ' the twist is spatially compensated and the isophase χ is no more twisted. Mathematically we can express this by saying that $\gamma \to \gamma'$ in such a way that

$$\chi = \theta' + \theta_{tw} \tag{8.73}$$

with $\theta' = \theta - \theta_{tw}$. Therefore, since in γ' the total phase is the unperturbed one, γ' must stay on a isotwist surface relative to the fixed initial unperturbed phase θ, because we measure no twist at all in the new configuration. This isotwist phase surface is given by a ribbon bounded by γ and with the other edge given by γ'. The frame that defines the ribbon is the unit vector $\hat{U}_0$ that initially stays on the unperturbed isophase $\theta = const.$.

In the second configuration $\hat{T}$ moves in such a way to make θ_{tw} vanish, or at least constant, keeping θ constant. The new vortex γ' will end up on the isophase $\chi = \theta = const.$. The net effect is that the total phase is unperturbed. Therefore γ' stays again on a isotwist surface with the given initial phase θ. This time the defect has ended up to stay again on an isotwist surface, but through a different mechanism that minimizes directly the twist phase without any reference phase compensation as in the first mechanism. Mathematically this can be expressed by saying that

$$\chi \to \theta \ \ as \ \ \theta_{tw} \to 0. \tag{8.74}$$

In this case we have made the value of the twist phase tend to zero without loss of generality: any constant term would make $\boldsymbol{\nabla}\theta_{tw} = \mathbf{0}$ along γ'. Here the vortex evolves modifying directly its twist while staying on a surface of

constant θ, while it is modifying the isotwist surface. If any isotwist, for any fixed, different initial phase θ, is a different ribbon, then in this case the defect evolves lying on an unperturbed, untwisted isophase (think of it as a flat surface bounded by the vortex), rotating any ribbon surface until everyone will coincide with the $\theta_{tw} = 0$ ribbon surface.
Both these two methods minimize $\nabla^2\theta_{tw}$ and $\mathbf{u}_{tw}$ along γ' but with two different consequences on the energy of the system. Indeed, the energy $E \propto |\boldsymbol{\nabla}\chi|^2 = |\boldsymbol{\nabla}\theta|^2 + |\boldsymbol{\nabla}\theta_{tw}|^2$. A transformation $\boldsymbol{\nabla}\theta \to \boldsymbol{\nabla}\theta - \boldsymbol{\nabla}\theta_{tw}$ gives an energy $E \propto |\boldsymbol{\nabla}\theta|^2 + 2|\boldsymbol{\nabla}\theta_{tw}|^2$ and the twist phase does change. Instead if $\boldsymbol{\nabla}\theta_{tw} \to 0$ directly, then $E \propto |\boldsymbol{\nabla}\theta|^2$, thus minimizing both real and imaginary parts of the energy. When the entire vortex γ ends up to stay on the 0-tw surface the vortex will be stable. Now we demonstrate that a minimization of θ_{tw} given by directly letting tend it to a constant value ($\boldsymbol{\nabla}\theta_{tw} = 0$), making coincide all the different isotwist surfaces, produces a new configuration for the vortex, γ', that has a non-zero writhe.

8.10.2 Demonstration of writhe production

Imagine to draw the closed defect γ in a parameter space that we call *twist space*, where any isotwist surface is parallel and foliates the space itself, and so the 0-twist surface does not coincide with the $2n\pi$-twist surface, for any $n \in \mathbb{Z}$. This space plays the role of a parameter space. In this space the initial and final points, $s = 0$ and $s = L$, of γ do not coincide and all the twist is in the form of torsion (this space is constructed so that all the isotwist surfaces are parallel leading to a zero intrinsic twist). Now, minimizing θ_{tw} leading all the isotwist surfaces to coincide means pulling back the curve on the first isotwist, thus extinguishing all the torsion. In this space, pulling back to compensate the torsion means counter-rotate $\hat{T}$ by a certain angle at any point of γ in real space. As a second step to close the curve we make the $\hat{T}$ at the ending point coincide with that at the initial one, $\hat{T}(L) = \hat{T}(0)$, hence closing γ smoothly. Anyway we can reverse the two movements first

of all closing the curve by pulling back the ending point until it coincides with the initial one (this is done by a rotation of $\hat{T}$ in real space) and then counter-rotating to extinguishing the extra torsion yet present (the total twist in real space). This would lead γ to stay completely on a unique isotwist surface, the 0-twist surface. We will follow this second possibility. This corresponds to two rotations of $\hat{T}$ in real space: one around the axis that connects any $\hat{T}$ with $\hat{T}_0$, that can always be chosen to be the z-axis, and the second one is a rotation of $\hat{T}$ relatively to the extinction of the residual torsion in the parameter space, hence equal to minus the extra torsion. Therefore we have two consecutive rotations. If we could demonstrate that the two angles summed in these two consecutive rotations have the form of those in (6.27) we would have demonstrated that during this movements a non vanishing amount of solid angle would be spanned by $\hat{T}$, if this must minimize the phase, and this would amount exactly to the writhe of γ. We first make the calculation in the parameter space and then explain how this translates in the real space.

Consider any tangent vector $\hat{T}$ in the parameter space introduced above. Call $\hat{T}_0$ the initial, fixed tangent to γ. Let us start by the first angle. To close the curve $\hat{T}$ must rotate around the z-axis that connects $\hat{T}$ and $\hat{T}_0$. Thus, for continuity, any $\hat{T}$ along the curve will rotate a bit around $\hat{z}$ (that is kept fixed). Therefore the first angle coincides with $\theta_{\hat{T}}(\hat{z})$ defined in (6.22). Now, to extinguish all the accumulated torsion we must counter-rotate $\hat{T}$ in real space by an angle which equals that torsion in twist space. First of all we see that, under the preceding rotation, the curve has changed its curvature and torsion. In particular, by Frenet-Serret formulas applied to the new rotated $\hat{T}$ we have

$$\frac{d}{d\,s}\hat{T} = c\hat{B} \times \hat{T}, \tag{8.75}$$

where $\hat{B}$ is the bi-normal and c the curvature. Hence, recalling (6.19),

$$\boldsymbol{\omega} = k\hat{B}. \tag{8.76}$$

The torsion of the rotated tangent vectors is

$$\tau = \frac{\hat{T} \cdot \left(\frac{d}{d\,s}\hat{T} \times \frac{d^2}{d^2\,s}\hat{T} \right)}{k^2}. \tag{8.77}$$

After substitution of (6.19) and (8.76) into (8.77), and some simple vectors algebra, we arrive at the following formula for the torsion:

$$\tau = \frac{\hat{T} \cdot \left(\boldsymbol{\omega} \times \frac{d}{d\,s}\boldsymbol{\omega} \right) + \hat{T} \cdot \boldsymbol{\omega} \left(\omega^2 - \left(\hat{T} \cdot \boldsymbol{\omega} \right)^2 \right)}{\boldsymbol{\omega}^2}. \tag{8.78}$$

After some calculation we arrive at

$$\tau = \hat{T} \cdot \boldsymbol{\omega}. \tag{8.79}$$

Finally we have that the torsion of the new curve is

$$\tau = (\hat{T} \cdot \hat{z})\, \omega = (\hat{T} \cdot \hat{z})\, \frac{\hat{z} \cdot \left(\hat{T} \times \frac{d}{d\,s}\hat{T} \right)}{1 - (\hat{z} \cdot \hat{T})^2}\, \hat{z}. \tag{8.80}$$

This is exactly the same term of (6.14). Hence the total torsion accumulated, and then the second angle, is

$$\tau_{tot} = \int (\hat{T} \cdot \hat{z})\, \omega = (\hat{T} \cdot \hat{z})\, \frac{\hat{z} \cdot \left(\hat{T} \times \frac{d}{d\,s}\hat{T} \right)}{1 - (\hat{z} \cdot \hat{T})^2}\, \hat{z}\, d\,s. \tag{8.81}$$

This correspond with the twist angle $\theta_{\hat{z}_\perp}(\hat{T})$ of (6.10). Therefore, the total angle of rotation of the closed defect to stay entirely on the 0-isotwist is the writhe:

$$\theta_{\hat{T}}(\hat{z}) - \tau_{tot} = \theta_{\hat{T}}(\hat{z}) - \theta_{\hat{z}_\perp}(\hat{T}) = 2\pi(1 + Wr). \tag{8.82}$$

This equation coincides with (6.27).

Two examples

To simplify the visualization of the mechanism let us consider two examples. The first is a twisted vortex ring γ with growing longitudinal velocity imposed by twist, i.e. growing $|\boldsymbol{\nabla}\theta_{tw}|$. This means that the rotation velocity of any fixed local frame $\hat{U}$ grows along the ring, keeping coincident the initial and final $\hat{U}$ (closed ribbon). It is simple to see that $\boldsymbol{\nabla} \cdot (|\boldsymbol{\nabla}\theta_{tw}|\, \hat{\nabla}\theta_{tw}) > 0$.

The same holds for a decreasing modulus of the gradient along γ. For any closed surface centered at one point of γ the net flux of $\boldsymbol{\nabla}\theta_{tw}$ along the curve (i.e. in the $\hat{T}$ direction) is positive. The vortex becomes unstable, the greater the Laplacian the greater the destabilization. Where the flux is maximum (at the point of injection where the ribbon closes) the vortex ring will be maximally deformed until it evolves in a new vortex γ' with non-zero writhe. After this transition there will be no more twist on the vortex and it will become stable evolving with a new GPE. From that time on the evolution will be reversible because the energy will be no more complex.

The second example consists of a twisted vortex ring with constant twist velocity magnitude and propagating in two opposite directions from a starting point on the ring. Locally the Laplacian is different from zero, but the total flux of the curve through any closed surface containing it is now zero, even for the unit sphere at infinity. In this case there is a source ($\nabla^2\theta_{tw} > 0$), where the twist is injected, and a sink ($\nabla^2\theta_{tw} < 0$), where all the twist phase gradient accumulates. Hence locally there will be a deformation and a consequent positive and negative writhe formation respectively, but as a whole, far from the filament, the total flux of the new tangent will be zero for any surface, and so will be the total writhe.

Chapter 9

Conclusions

To provide a full, hydrodynamic interpretation of the evolution of defects when twist is injected we have derived a new, complete set of equations governing the evolution of density, phase and twist. This has been done by applying the defect gauge theory developed by [45] and by implementing the results of [65] to determine the correct vorticity transport equation. The full set of hydrodynamic equations (8.43) in the fundamental variables ρ, u_0 and $\theta_{\rm tw}$, supplemented by the definition $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ and the vorticity equation (8.42) (9 equations in 9 variables) is complete. These equations establish a correspondence between the mean field theory of quantum fluids and the standard treatment of classical fluid mechanics. The set of governing equations thus derived can be readily implemented in numerical simulations, providing a complementary approach to explore topological and dynamical features of quantum fluids using classical hydrodynamics methods.

In this respect eqs. (8.43) allow exploration and identification of actual scenarios by investigating the consequences of superposed twist through the analysis of twist energy relaxation.

Moreover, by analyzing the associated Hamiltonian we have demonstrated that the dynamics is non-Hermitian (Lemma 1) and by computing total energy we have shown that the system can be linearly stable or unstable according to twist diffusion. We have shown that a twisted vortex is unstable if the twist phase has a Laplacian greater than zero, as demonstrated also

in [67]. Instability in non-Hermitian systems has already been explored in other contexts, for instance in [81].

Then we have explained what happens when the system becomes unstable. We have shown that two possibilities can happen: if the twist phase is global and singular at a central point, i.e. if the perturbation of an isophase is extended to all the isophase, with the exception for the center, the dynamics is dominated by a topological phase: to minimize the energy and stabilize the vortex a new, central phase defect will form, as already shown by theoretical arguments [54]. All the extra energy due to twist is spent to create this singularity. The particular case of a vortex ring with uniform twist has been considered, showing that indeed the twisted case has higher energy than the zero twist case. In the case examined the axial symmetry of the initial vortex ring and the uniform superposed twist triggers the immediate production of a new, central defect with simultaneous formation of weak oscillations of the density profile (visualized by the corrugation of the density level in Figure 8.1c). As pointed out, production of new defects can be seen as a manifestation of the celebrated Aharonov-Bohm effect [42]. We have also proposed an experiment to verify this secondary vortex production. This result is in good agreement with the observed incipient instability and the subsequent development of a new defect [12]. An analytical proof of this phenomenon relies on the fact that for such systems total helicity is known to be conserved, remaining identically zero during evolution and through topological changes [9, 55, 77].

However, in general new defect production is not necessarily a consequence of twist relaxation. Configurational changes that alter the geometry of defects in space can also occur. Indeed we have shown that for a local phase perturbation no singularity will form. The defect line is unstable and changes in time. It must change its spatial configuration in order to change the total phase so to compensate the destabilizing twist, minimizing both the imaginary and real parts of the energy. The new configuration γ' stays on an isotwist surface, which in space is a ribbon with edges γ and γ'. There are

two ways to minimize the energy. The best one for minimizing the energy is when γ' evolves onto the unperturbed isophase $\theta = const.$ and rotate all the isotwist surfaces to make them coincide with the 0- twist surface. This mechanism produces a new, more stable state in which the defect line has non-zero writhe and no more twist. In this case the evolution is due to a geometric phase that turns into writhe. The phenomenon that changes the vortex configuration producing writhe from the twist is irreversible, because of the initially non-zero imaginary part of the energy. This means that a coiled quantum vortex in a BEC cannot automatically twist unless some external field twists it. When writhe is formed the state is changed and the equation of motion becomes the usual GPE since $\boldsymbol{\theta}_{tw}$ has been minimized. The evolution of a coiled defect line is reversible because $\boldsymbol{\nabla}\theta_{tw} = \mathbf{0}$.

It would be interesting for the future to further explore and study the time dependence of θ_{tw} in order to see in what way it restores the Hermiticity of the entire system. Part of the work of further studying the stability mechanism has been done in [82]. Moreover, it would be of great interest for the study of twisted vortex dynamics to see a possible role of the twirl concept as defined in [57], that is related with defects of high-order derivatives of the order parameter.

The mechanisms that occur for twisted quantum defects open a new scenario on the possible relations between the dynamics of quantum vortices and their topological and geometric properties.

Appendix A

Fundamentals of Quantum Mechanics

In quantum mechanics physical observables O, like energy, linear and angular momentum, are modeled by operators $\hat{O}$ on a Hilbert space $\mathcal{H}$. The state of a system is a vector $|\psi\rangle$ of $\mathcal{H}$. Any operator acts linearly on a state of the Hilbert space giving a new state: $\psi' = \hat{O}\,|\psi\rangle$. The quantities measured must be real numbers and are represented by the eigenvalues of these operators. Moreover, any measurement of an observable is mathematically described by a complete set of orthogonal states that are eigenstates of the measured observable. Therefore operators that represents physical observables must be self-adjoint.

Definition A.0.1. Let $\mathcal{H}$ be an Hilbert space. Let $\hat{O}$ an operator acting on a Hilbert $\mathcal{H}$. Let O^+ be its adjoint operator. We say that O is *self-adjoint* if $\hat{O} = \hat{O^+}$ or, equivalently, if $\forall\,|\psi\rangle\,, |\xi\rangle \in \mathcal{H}$ it holds that $\langle\hat{O}\xi|\psi\rangle = \langle\xi|\hat{O}\psi\rangle$, where $\langle .|.\rangle$ is the inner product in $\mathcal{H}$. For finite dimensional Hilbert spaces a self-adjoint operator is called *Hermitian*.

Definition A.0.2. A state $|\psi\rangle$ is *normalized* if $\langle\psi|\psi\rangle = 1$.

We now recall the probability amplitude and the wave function concepts in quantum mechanics.

Remark. Given two normalized states $|\xi\rangle$ and $|\psi\rangle$, the quantity $\langle\xi|\psi\rangle$ rep-

resents the *probability amplitude* that a system initially in the state $|\psi\rangle$ is found in the state $|\xi\rangle$ after the measure. The quantity $|\langle\xi|\psi\rangle|^2 < 1$ is a non-negative real number and represents the *probability* of passing from the state $|\psi\rangle$ to the state $|\xi\rangle$ after a measure. Probability, instead of probability amplitude, is a measurable quantity.

Definition A.0.3. Given the eigenstates of the self-adjoint position operator $\hat{\boldsymbol{X}}|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle$, a wave function $\psi(\boldsymbol{x})$ is the projection of a state $|\psi\rangle$ onto the eigenstates $|\boldsymbol{x}\rangle$ and is defined

$$\psi(\boldsymbol{x}) := \langle\boldsymbol{x}|\psi\rangle\,. \tag{A.1}$$

This means that a wave function represents the components of a state with respects to a position measure. Since $|\psi\rangle$ has a finite norm, $\psi(\boldsymbol{x})$ is an L^2-complex valued function representation of the state $|\psi\rangle$. It represents the probability amplitude that if the system is in a state $|\psi\rangle$ it is found in the position $\boldsymbol{x}$. The term $|\psi(\boldsymbol{x})|^2$ is the *probability density*. An Hermitian operator $\hat{O}$ is always diagonalizable and its eigenstates form a complete set of orthonormal eigenstates, i.e. a set of eigenstates $|o\rangle$ such that $\hat{O}|o\rangle = o|o\rangle\,, \quad \langle o|o\rangle = 1$.

Remark. Let $|o\rangle$ be a continuous set of orthonormal eigenstates of a self-adjoint operator $\hat{O}$. Then

$$\int |o\rangle\langle o|\,d\,o = id_{\mathcal{H}} \tag{A.2}$$

where $d\,o$ is the measure defined in the continuous set of eigenstates.

Measuring position means to know $|\langle\boldsymbol{x}|\psi\rangle|^2 = |\psi(\boldsymbol{x})|^2$. Since the probability density is a squared modulus, this means that one completely loses any information about the phase $e^{i\phi}$ associated with the complex wave-function.Therefore, we have the following

Remark. If two states $|\psi\rangle$ and $|\xi\rangle$ differ only by a phase, $|\psi\rangle = e^{i\phi}|\xi\rangle$ then they represent the same physical state, because their respective probability densities are equal for equal positions: $|\psi(\boldsymbol{x})|^2 = |\xi(\boldsymbol{x})|^2$. This means that there is a $U(1)$ gauge freedom in choosing the phase of a single state and

that the physical sector of the Hilbert space $\mathcal{H}$ is the *space of rays*, i.e. the projective Hilbert space

$$\mathcal{H}_{phys} := \mathcal{H}/U(1). \tag{A.3}$$

By inserting the identity resolution of equation (A.2) we can rewrite the probability amplitude fixing a coordinates system.

Remark. The probability amplitude for a system to be found in a state $|\xi\rangle$ being initially in a state $|\psi\rangle$ is

$$\langle\xi|\psi\rangle = \int \xi(\boldsymbol{x})^*\psi(\boldsymbol{x})d^3\boldsymbol{x}. \tag{A.4}$$

We can also rewrite the norm of a state $|\psi\rangle$:

Remark.

$$\langle\psi|\psi\rangle = \int |\psi(\boldsymbol{x})|^2 d^3\boldsymbol{x}. \tag{A.5}$$

This explicitly shows that $\psi(\boldsymbol{x})$ is a L^2-function and shows that, if the state is normalized, $|\psi(\boldsymbol{x})|^2 d^3\boldsymbol{x}$ represents the probability measure (in the basis of position).

A fundamental quantity in quantum mechanics is the expectation value of an observable, computed with respect to a given state.

Definition A.0.4. The expectation value of an operator $\hat{O}$ with respect to a state $|\psi\rangle$ is

$$\langle\hat{O}\rangle_\psi := \frac{\langle\psi|\hat{O}\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int o\,|\psi(o)|^2 d\,o}{\int |\psi(o)|^2 d\,o}. \tag{A.6}$$

The second equality holds by inserting the identity resolution (A.2) in the bases given by the eigenstates of $\hat{O}$. A transformation of the system must preserve the probability. Hence, any transformation is represented by a unitary operator $\hat{U}$ defined as follows.

Definition A.0.5. An operator $\hat{U}$ is *unitary* if

$$\hat{U}\hat{U^+} = id_{\mathcal{H}}. \tag{A.7}$$

We have the following important theorem.

Theorem 2 (Stone)**.** *Any one-parameter group of unitary operators* $\hat{U}_t$, $t \in \mathbb{R}$ *is in a one-to-one correspondence with a self-adjoint operator* $\hat{O}$ *by the exponential map:*

$$\hat{U}_t = e^{-\frac{i}{\hbar}\hat{O}\cdot t}. \tag{A.8}$$

The operator $\hat{O}$ is called the *generator* of the group of transformations $\hat{U}_t$ by a parameter t. For instance, in finite dimension, energy can be transformed in a Hermitian operator and is thought as the generator of the time evolution operator $\hat{S}_t$, where the parameter is time t:

$$\hat{S}_t = e^{-\frac{i}{\hbar}\hat{H}\cdot t}. \tag{A.9}$$

For Hamiltonian operators (from now on only Hamiltonians) which depends on time, we have:

$$\hat{S}_t = e^{-\frac{i}{\hbar}\int_0^t \hat{H}(t')\cdot dt'}, \tag{A.10}$$

i.e. the evolution from time 0 to time t is the product of the evolutions in the intermediate times.

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